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A stability-theory perspective to synchronisation of heterogeneous networks

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A stability-theory perspective to synchronisation over heterogeneous networks

Introduction and motivations

Perhaps the first idea that one associates to the keyword *network* is *telecommunication*, especially in a world of rapidly-evolving technology. However, the concept of networking is ubiquitous in other emerging applications, such as systems in social, medical and bio applications, not to mention energy transformation. These are not *just* large-scale and complex systems but they are characterised by decentralised, distributed, networked compositions of heterogeneous and (semi)autonomous elements. These new systems are, in fact, *systems of systems*.

The complexity of network-interconnected systems may not be overestimated. For instance, in energy-transformation networks, the improper management of faults, overloads or simply adding to, or subtracting a generator from, the transportation network may result in power outages or even in large scale (continent-wide) blackouts. In neuronal networks, experimental evidence shows that inhibition/excitation unbalance may result in excessive neuronal *synchronisation*, which, in turn, may be linked to neuro-degenerative diseases such as Parkinson and epilepsy.

Even though the nature of each constituting system and the interconnections among them differ drastically from one domain of study to another, at the level of mathematical and even philosophical abstraction, they all possess common features, share similar requirements and may be analysed via common approaches. What is more, we sustain that the behaviour of networked systems may be studied through two complementary paradigms. The first, most evident to the observer, is *synchronisation*. Generally speaking, this pertains to the scenario in which a group of interconnected systems coordinate their motions, in a mathematical sense –see [34]. Furthermore, synchronisation of networked systems (also referred to as *agents*) leads to a second, somewhat more abstract, paradigm: the appearance of a *coherent* behaviour. At least intuitively, the latter may be related to a global coordination of motions which stems from the local interactions among the components of the network.

Coherent, *emergent*, behaviour is not a mere conjecture; it is observed and studied within numerous scientific disciplines such as chemical and biological sciences, physics, sociology, physiology, complexity theory, systems sciences, philosophy of sciences, to name a few. Depending on the specific area of research this "new" behaviour is known under various aliases:

- collective behaviour;
- self-organised motion;
- emergence;
- synergy [32];
- cooperativeness, symbiosis, epistasis, threshold effects, phase transitions, co-evolution, heterosis, dynamical attractor ...

Along with the varied terminology, during the last few decades, research on this subject generated a vast, and not particularly congruent[1], amount of literature which at the conceptual level reaches little or no agreement neither regarding the definitions and analysis frameworks of these notions nor concerning the relationships among these notions. What is more, at a farther level of abstraction, there remains a long-lasting profound debate, on the essence of emergent behaviour itself ([79]), which, roughly, divides opinions into some which prone holistic paradigms and others in favour of reductionist ones.

Following [79], reductionism can be defined as the view that the best understanding of a complex system should be sought at the level of the structure, behaviour and laws of its components parts plus their relations. While the basic idea of emergionism is, roughly, the converse. The notion of emergence implies that even though emergent features of the whole are not completely independent of its parts, in some ways, the former transcends the latter.

Yet, the *interlaced* character of the two notions is a fundamental aspect which is important to underline, for this we quote P. Corning from *Holistic Darwinism: Synergy, Cybernetics, and the Bio-economics of Evolution*, University of Chicago Press, Aug 15, 2010 :

> "Nowadays *complexity* is also recognised to be a distinct *emergent* phenomenon that requires higher-level explanations. In fact, there is a rapidly growing literature in complexity theory – much of it powered by the mathematics of non-linear dynamical systems theory – which is richly *synergistic* in character; it is primarily concerned with *collective properties* and collective effects."

On the other hand, it is well agreed that the system has to reach a combined threshold of diversity, organisation, and connectivity before some coordinated or emergent behaviour appears. In the broadest terms all these notions may be viewed as the study of microscopic vs macroscopic organisation of a complex system. Or, following more mathematical terms, we single out the following issues which play a key role in analysis and control of networked systems:

- the coupling strength;
- the network topology;
- the type of coupling between the nodes, $i.e.$, how the units are interconnected;
- the dynamics of the individual units.

Based on this duality, *synchronisation-emergent-behaviour*, our objective is to develop a framework of study for systems interconnected over heterogeneous networks. We aim at characterising fundamental limitations and capabilities of networked systems in terms of network topology, link communication and nodes' dynamics, performance, $etc.$ The variety of realistic complications that such a framework must accommodate, such as communication delays, measurement uncertainty, competitive environments are clear evidence of the ambitious nature of this goal, however, these aspects are not addressed here. We focus or attention on nodes' dynamics and network topology.

We address the general problem of synchronisation from a stability view-point. We limit our study to the *analysis* paradigm, as opposed to that of *controlled* synchronisation. Our theoretical

[1] See J. Halley, D. A. Winkler's "Classification of emergence and its relation to self-organization". *Complexity*, 13(5), 10-15, (2008).

framework pertains to large numbers of systems which we consider to be interconnected over a network. However, we disregard technological and dynamical aspects related directly to the network itself. That is, our study focuses on *structural* properties of the network, which affect the synchronisation of the agents' motions in one way or another.

We approach the networked systems paradigm from a control-theory perspective because systems and control science provides a general framework adequate to handle models of dynamic physical, chemical, biological, economic and social systems by developing concepts and tools for their analysis and design. It integrates contributions from mathematics, signal processing, computer science and from many application domains. In addition, the notion of synchronisation is well defined in dynamic control theory –see $e.g.$, [12, 13, 15] and there are many tools issued both from the dynamic systems and automatic control domains for the synchronisation analysis of complex (networked) systems. Nonetheless, the situation is quite different concerning the analysis of synchronised emergent behaviour.

Thus, this memoir presents a sample of our research realised in the past ten years on analysis of synchronisation of nonlinear systems' over networks. The technical results that we present address the first aspect in the previous list: how the coupling strength affects synchronisation of systems interconnected over networks. We establish results on what is called *practical* synchronisation, $i.e.$, we give conditions under which the motions of heterogeneous systems asymptotically become "similar". The results that we choose to present have been obtained as a byproduct of the supervision of two recent PhD theses:

L. Conteville "*Analyse de la stabilité des réseaux d'oscillateurs non linéaires, applications aux populations neuronales*", Univ Paris-Sud, 17th Oct. 2013.

A. El-Ati, "*Synchronization analysis of a directed network of coupled heterogeneous nonlinear oscillators*", Univ Paris-Sud, Dec. 2014.

The material is organised in the two main technical chapters of the memoir: Chapter 2 on the thesis of L. Conteville and Chapter 3 on the thesis of A. El-Ati.

Even though we contribute with the basis of a solid *analysis* framework ours remains a modest contribution if compared to the vast area of research which includes a range of problems that remain open. Some of these are briefly discussed in Chapter 4 where we mention research directions that may lead to the supervision of new PhD-thesis and other projects.

For the sake of clarity and to render the document self-contained, we start our technical exposition with an introductory chapter on general aspects of synchronisation and oscillations.

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CHAPTER 1

On synchronisation and oscillations

1 Synchronisation

The study of synchronisation has been the subject of research in several disciplines before control theory: it was introduced in the 1970s in the USSR in the field of mechanical vibration by Professor Blekhman. Ever since, research on synchronisation has been popular among physicists, $e.g.$, in the context of synchronisation of chaotic systems since the early 1990s, but also among engineers, especially on automatic control, in the context of synchronisation of network-interconnected systems. Recent applications of controlled synchronisation involve various areas of science and technology, –see, $e.g.$, [41, 7, 77, 51, 58, 83, 1].

As its etymology suggests, synchronisation may be defined as the adjustment of rhythms of repetitive events (phenomena, processes, ...) through weak interaction. In that regard, let us underline the following concepts which are often used in the physics literature and are related to synchronisation:

Repetition of a process or an event. We consider the synchronisation of phenomena that occur independently of each other, $i.e.$, by their own "force", and which repeat themselves through time. Let us stress that *repetition* does not imply periodicity (events repeat themselves every exact constant amount of time measures) but it indicates that the processes happens over and over again, regularly.

The **rhythm** marks the pace of the repetition of a process. That is, two processes are synchronised if they have the same rhythm. To fix the ideas, one may think of two pendulum clocks oscillating with different frequencies and out of phase; each with its own cadency. If the pendula are decoupled physically, that is, if the movement of one does not exert any influence on the movement of the second whatsoever, the clocks will continue oscillating at their own pace. Instead, the clocks may adjust their rhythms one with respect to the other if there exists a-

weak interaction, $i.e.$, a physical coupling. For instance, two pendulum clocks hanging from a beam will transmit each other vibrations of very small intensity through the beam. As a result, and this depending on other factors, the two clocks may start to oscillate at the same frequency, after some time.

One can identify various types of synchronisation. Firstly, one can make the distinction between self and controlled synchronisation.

Self-synchronisation pertains to the case when two or more oscillators interact through a weak coupling without external stimuli and they influence each other's rhythm until they attain a synchronised *motion*[1]; an adjustment of rhythms.

Controlled synchronisation pertains to the case when via external stimuli, two or more systems are forced to enter in synchrony.

Secondly, depending on its nature, two forms of synchronisation may be distinguished:

Mutual synchronisation pertains to the case in which two or more processes adjust their rhythms with no particular priority on either one's rhythm;

master-slave synchronisation consists in one system imposing its own rhythm of motion to the second. The dominant process is commonly called the *master* and the entrained system is called the *slave*.

Self-synchronisation may be observed in a number of natural phenomena. As a matter of fact, as soon as the universal (nonetheless abstract) concept of $time$ enters into the equation one immediately can make a link to the concept of time-keeper, $clock$. From here to synchrony is but a small step. Endless examples of synchronised clocks may be found in nature: the movements of planets, the circadian rhythm synchronised with the day/night cycle, the heart pace maker cells, neuron firing, *etc*.

Other situations that relate to the concept of synchronisation are group formations of animals such as banks of fish, flocks of migrating birds, etc. Such natural phenomena have inspired engineers and scientists to study both, self and controlled synchronisation, for a number of decades and within a variety of domains including (nonlinear) physics, biology, medicine, mechanical engineering, mechatronics, robotics, computer science, etc.

Recognising that synchronisation consists in the *adjustment* of rhythms due to local interaction leads us naturally to put this paradigm in a mathematical context suitable for analysis from a control-systems perspective. Let us focus on a group of N nonlinear systems which, for simplicity, are assumed to be acceptably modelled by ordinary differential equations, $i.e.$,

$$\begin{array}{rcl}\dot{x}_1 & = & F_1(t, x_1) + G_1(t, x_1, \ldots, x_N) \qquad (t, x_i) \in \mathbb{R} \times \mathbb{R}^n, \quad i \in 1, \ldots, N \\ & \vdots & \\ \dot{x}_i & = & F_i(t, x_i) + G_i(t, x_1, \ldots, x_N) \\ & \vdots & \\ \dot{x}_N & = & F_N(t, x_i) + G_N(t, x_1, \ldots, x_N)\end{array}$$

[1]It may be convenient to specify that by *motion*, we refer to the definition recalled, from Soviet literature, by Hahn in [34] that is, for a differential equation $\dot{x} = f(t, x)$, the curve described by the points (t, x) in the motion space $\mathbb{R}_+ \times \mathbb{R}^n$. In particular, a trajectory $t \mapsto x$ is a projection of the motion into the phase space $\mathbb{R}^n$.

with drifts F_i and interconnections G_i. Then, defining $Q : \mathbb{R} \times \mathbb{R}^{nN} \to \mathbb{R}^n$ as a continuous function, we say that the systems are synchronised with respect to Q if

$$\lim_{t\to\infty} Q(t, \boldsymbol{x}(t)) = 0, \quad \boldsymbol{x} := [x_1 \cdots x_N]^\top.$$

The latter is a fairly general way of stating the synchronisation paradigm[2] whether it is of the master-slave or the mutual type. It concerns couplings of dynamic systems which may be forced or unforced. In a number of situations the significant synchronisation problem may concern the whole state then, Q may be defined as

$$Q(t, \boldsymbol{x}) := \begin{bmatrix} x_1 - x_2 \\ \vdots \\ x_{N-1} - x_N \end{bmatrix}.$$

Master-slave synchronisation pertains to the case in which a slave system, say described by the dynamical equation

$$\dot{x}_s = f(t, x_s, x_m) \quad x_s(t_\circ) \in \mathbb{R}^n, \; t \geqslant t'_\circ \geqslant 0 \tag{1.1}$$

is synchronised with a master system

$$\dot{x}_m = f(t, x_m) \quad x_m(t_\circ) \in \mathbb{R}^n, \; t \geqslant t_\circ \geqslant 0 \tag{1.2}$$

in the sense that the master system performs a free motion whereas the slave system must follow the movement of the master's. Hence, we say that the systems are synchronised if for any initial conditions $t_{m\circ}$, $t_{s\circ}$, $x_{m\circ}$, and $x_{s\circ}$ the respective state trajectories of systems (1.2) and (1.1) satisfy:

$$\lim_{t\to\infty} |x_s(t) - x_m(t)| = 0. \tag{1.3}$$

The popular tracking control problem, broadly addressed in the literature, can be seen as a master-slave synchronisation problem –see, *e.g.*, [26].

In the case of mutual synchronisation, the systems involved must coordinate their motions in consequence of their local interactions. In control and stability theory mutual synchronisation is an especially popular research area partially driven by the *consensus* paradigm.

2 Consensus

Generally speaking, consensus pertains to the case in which a set of agents acquire a synchronised motion and they all reach a common steady motion. One of the main features is that the motion of no agent has priority over that of any other one.

Thus, the consensus problem consists in establishing conditions under which the differences between any two motions among a group of dynamic systems, converge to zero asymptotically. This, under the assumption that there exist (weak) interactions among the dynamical systems.

[2] Further generalisations of this notion may be found in [12].

We may cite a number of possibilities for the systems' motions which lead to perfect mutual synchronisation; the simplest case, is that in which all the motions converge to a common *equilibrium* point. That is, all trajectories converge to the same *constant* value, which depends on the initial conditions. We may also consider the case in which all motions converge to a common one which does not necessarily correspond to any of the systems'. For instance, consider a group of *oscillators* which all describe the same sinusoidal trajectory along time, possibly in phase but not necessarily and which does not correspond to the natural trajectory described by any of them, when being decoupled. From a geometric viewpoint, one may consider the problem of a group of planes flying in a *formation* hence, not all of them describe the same trajectory, nor they follow the same path but a constant relative position from one another is kept all along their paths.

Having these illustrations in mind it seems natural to be interested in the basic *consensus problem*, motivated by technological applications such as formation control of vehicles (marine, aerial, terrestrial) and teleoperation [14, 33, 8, 65] but also by other disciplines where the study of consensus has an impact, such as neuroscience [35, 30, 20] and telecommunications.

The simplest case of consensus is illustrated via driftless systems, $i.e.$, simple integrators

$$\dot{x}_i = u_i \qquad i \in \{1, \ldots, N\}.$$

It is assumed that each agent communicates with a neighbour, with several nodes or with none. The state of each agent corresponds to the "information" that one agent has at any instant. It is desired to ensure that all agents reach consensus on the information that is, all states of the system of agents represented by

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij}(x_i - x_j), \qquad a_{ij} \geqslant 0, \tag{1.4}$$

are required to converge and stabilise at the same equilibrium. Following the notation of Graph Theory, the interconnection coefficients, a_{ij}, are positive if any information flows from the jth (parent) node to the ith (child) node; obviously, $a_{ii} > 0$. The matrix composed of such coefficients is the so-called adjacency matrix. Then, the resulting dynamics of the interconnected integrators is given by the linear differential equation

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_N \end{bmatrix} \quad \Longrightarrow \quad \dot{x} = -Lx$$

where L is the corresponding *Laplacian*, defined as

$$L = \begin{bmatrix} \sum_{i=2}^{N} a_{1i} & -a_{12} & \cdots & -a_{1N} \\ -a_{21} & \sum_{i=1, i\neq 2}^{N} a_{2i} & \cdots & -a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{N1} & -a_{N2} & \cdots & \sum_{i=1}^{N-1} a_{Ni}, \end{bmatrix}.$$

We say that the systems reach consensus if and only if $x(t) \to x^*$ as $t \to \infty$ where x^* is a common equilibrium point.

In the last decade or so there has been an explosion of literature in the control systems community, on the problem of consensus. While many interesting problems have been solved, such literature mainly concerns natural extensions of an almanac of results on tracking control using various classical techniques (adaptive control, sliding-mode control, observer-based, $etc.$) in classical settings (nonlinear, hybrid, Lagrangian, Hamiltonian, $etc.$). See, $e.g.$, [73].

To transcend, it is convenient to recognise that consensus over networks is a problem that appeals to

- dynamical systems theory (solutions of differential equations),
- stability theory (for the equilibrium x^*),
- graph theory,
- numerical methods,

among other disciplines.

In this memoir we present a general framework for the study of consensus of heterogeneous systems interconnected over networks. We introduce an original and general concept which we call *dynamic consensus*, which generalizes the concepts of consensus with respect to an equilibrium, or even with respect to a synchronisation manifold. Our research in the area is especially motivated by the synchronisation paradigm of *oscillators* in a broad sense therefore, we find it appropriate to present a brief survey on this subject.

3 Oscillations

Most naturally, we think of an oscillation as a periodic motion say, sinusoidal. However, this is an oversimplification of the concept. Indeed, periodicity is sufficient but not necessary; it is a particular case of *repetitiveness*. Therefore an oscillation may be thought of as the repetition of a pattern. If we think of the motion of a point through space we refer to it as oscillatory if the point keeps passing by the same position (or arbitrarily close to it) infinitely many times. Furthermore, oscillations are characterised by a *rhythm*.

In addition, in the study of oscillators (systems which produce oscillatory motions) in the context of synchronisation it is important to distinguish between *forced* and free *self-sustained* oscillators. Forced oscillators owe their particular motion due to external stimuli. Self-sustained oscillations are due only to the structure of the system and its parameters; the fundamental feature is that when taken away from its environment, a self-sustained oscillator continues to oscillate at its own pace, as if there existed a perfect balance between the energy "injected" in the system and that "consumed" $i.e.$, dissipated. After a perturbation vanishes, a self-sustained oscillator recovers *by itself* its own rhythm.

Then, from a mathematical modelling viewpoint, as we pretend to study oscillations as a type of behaviour of *dynamical systems* it seems natural to model the latter via differential equations. The simplest (self-sustained) oscillators that one can imagine are defined by linear autonomous equations

$$\dot{x} = Ax$$

where A has all its eigenvalues on the imaginary axis. If the system is of second order $i.e.$, $n = 2$ the solutions correspond to harmonic oscillations. For higher order systems, the solutions will involve more than one frequency, leading to quasi-periodic oscillatory solutions. Periodic solutions form closed curves in the phase space, called closed orbits. In the case of second-order systems, we call a closed orbit a **limit cycle** if there exists no other closed trajectory in its neighbourhood.

Some well-known examples of self-sustained oscillators, which are used as case-study in this document, are described next.

3.1 The generalized Stuart-Landau oscillator

The Stuart-Landau equation, which represents a normal form of the Andronov-Hopf bifurcation, is given by

$$\dot{z} = -\nu |z|^2 z + \mu z \tag{1.5}$$

where $z \in \mathbb{C}$ denotes the state of the oscillator, ν, $\mu \in \mathbb{C}$ are parameters defined as $\nu = \nu_\mathsf{R} + \mathrm{i}\nu_\mathsf{I}$ and $\mu = \mu_\mathsf{R} + \mathrm{i}\mu_\mathsf{I}$. The real component of μ, μ_R, determines the distance from the Andronov-Hopf bifurcation. In the literature, the system (1.5) with $\mu_\mathsf{R} > 0$, is known as the Stuart-Landau oscillator [6], [46], [61]. It is also known as the Andronov-Hopf oscillator [68]. The Stuart-Landau equation is in normal form, which means that the limit cycle dynamics of many other oscillators can be transformed onto or can be approximated by the dynamics given by Equation (1.5), see [39]. We cite, for example, the papers [36], [83] where the van der Pol oscillator and the Haken-Kelso-Bunz (HKB) model in the neuro-physiological applications are approximated by Equations (1.6a) and (1.6b).

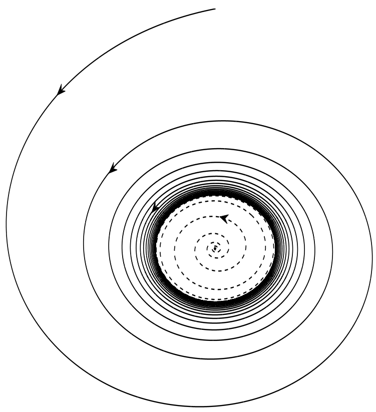


Figure 1.1: Trajectories of the Stuart-Landau oscillator on the complex plane. If $\mu_\mathrm{R} > 0$ the origin is unstable but all trajectories tend to a stable limit cycle with radius $r = \sqrt{\frac{\mu_\mathrm{R}}{\nu_\mathrm{R}}}$

The behaviour of the Stuart-Landau oscillator, which is illustrated in Figure 1.1, may be analysed in polar coordinates. To that end, let $z = re^{i\varphi}$ then, the equations for the radial amplitude r and the angular variable φ can be decoupled into:

$$\dot{r} = \mu_{\mathsf{R}} r - \nu_{\mathsf{R}} r^3 \quad (1.6a)$$

$$\dot{\varphi} = \mu_{\mathsf{I}} - \nu_{\mathsf{I}} r^2. \quad (1.6b)$$

When $\mu_{\mathsf{R}} < 0$, Equation (1.6a) has only one stable fixed point at $r = 0$. Moreover, the latter is Lyapunov (globally exponentially) stable. However, if $\mu_{\mathsf{R}} > 0$, this equation has a stable fixed point $r = \sqrt{\frac{\mu_{\mathsf{R}}}{\nu_{\mathsf{R}}}}$, while $r = 0$ becomes unstable. This implies, in this case, that the trajectories of the system converge to a circle of radius r, starting from initial conditions either inside or outside the circle. Thus, the latter is an attractor and the system (1.5) exhibits periodic oscillations. In this case, z represents the position of the oscillator in the complex plane and $z(t)$ has a stable limit cycle of the amplitude $|z| = \sqrt{\frac{\mu_{\mathsf{R}}}{\nu_{\mathsf{R}}}}$ on which it moves at its natural frequency. The bifurcation of the limit cycle from the origin that appears at the value $\mu_{\mathsf{R}} = 0$ is known in the literature as the Andronov-Hopf bifurcation and the curves

$$\Gamma_\alpha = \sqrt{\frac{\mu_{\mathsf{R}}}{\nu_{\mathsf{R}}}} \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} \quad (1.7)$$

define the limit cycle of the system.

Last but not least, so-called *chaotic* systems are complex oscillatory systems widely studied in the literature of Nonlinear Physics; they deserve a more detailed description, which we provide next.

3.2 Chaotic Systems

These are systems which have an oscillatory behaviour in a very broad sense. Motions of chaotic systems are certainly not periodic but they are recurrent. The solutions to differential equations which model a chaotic system present high sensitivity to initial conditions hence, solutions starting arbitrarily close to each other will tend to separate exponentially. By the same property, solutions pass arbitrarily close to any point infinitely many times. When plotted against time, the solution of a chaotic system is seen to oscillate in a seemingly random manner however, chaotic systems are deterministic and are often represented by ordinary differential equations (in continuous time) or by difference equations (in discrete time). Chaotic systems' models are often autonomous (self-sustained oscillators) and of at least, 3rd order[3]. As it is characteristic of oscillators, chaotic solutions are bounded but they contain an unstable equilibrium which repels all solutions towards a strange attractor.

A more precise definition is the following.

Chaotic system. Let $\mathcal{A}$ be a topologically transitive attractor for the trajectories of the system

$$\dot{x} = F(x), \qquad x \in \mathbb{R}^n$$

> The set $\mathcal{A}$ is called strange attractor if it is bounded and all solutions which start from $\mathcal{A}$ are Lyapunov unstable. The system is called chaotic if it has at least one strange attractor.

[3]Although second order systems such as the van der Pol oscillator may be induced into chaotic behaviour by the action of a simple periodic input.

Local instability is one of the important features of a chaotic system and it may be analysed using the so-called Lyapunov characteristic exponents, λ. These measure the degree of sensitivity in the initial conditions. More precisely, consider a map $f : \mathbb{R}^n \to \mathbb{R}^n$ and its Jacobian

$$\nabla f(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & \cdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

at the point x. Let υ_i be a tangent vector at the point x. Then,

$$\lambda(x, \upsilon_i) = \lim_{t\to\infty} \frac{1}{t} \ln |\nabla(x)\upsilon_i|$$

The so-called Oseledec theorem states that this limit exists for almost all points x and all tangent vectors υ_i. Note that there are at most n distinct values of λ as there are n tangent vectors υ_i. The numbers λ_i are the Lyapunov exponents at x. For linear time-invariant systems the Lyapunov exponents coincide with the real parts of the eigenvalues of the system. If a system is chaotic it has at least one positive Lyapunov exponent, however, in general, having positive Lyapunov exponents does not imply chaotic behaviour.

Celebrated examples of chaotic systems include the Lorenz oscillator [55] but the literature on physics is abundant in more recent models, such as the so-called Lü system [59], which, to some extent, generalizes the former.

The Lorenz system

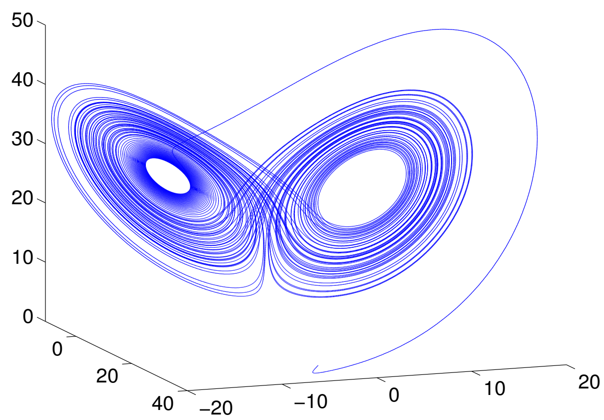


Figure 1.2: Phase portrait of $x_1(t)$ *vs.* $x_2(t)$ for the the Lorenz system with $\sigma = 10$, $\beta = 8/3$, $\rho = 25$.

The Lorenz system is probably the best-known of chaotic oscillators. It was discovered as a model for weather prediction by E. N. Lorenz. The equations are

$$\begin{aligned}\dot{x} &= \sigma(y-x)\\ \dot{y} &= \rho x - y - xz\\ \dot{z} &= xy - \beta z\end{aligned}$$

where σ is called the Prandtl number and ρ is called the Rayleigh number. All parameters σ, ρ and β are positive. Often, σ and β are fixed for instance at $\sigma = 10$ and $\beta = 8/3$ and ρ is left as a free parameter to tune.

The linearisation of the Lorenz system around the origin yields

$$\begin{aligned}\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} &= \begin{bmatrix}-\sigma & \sigma\\ (\rho-1) & -1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}\\ \dot{z} &= -\beta z\end{aligned}$$

The third equation is obviously exponentially stable at the origin. The first two equations form a second-order linear time-invariant system with poles at the roots of the characteristic polynomial:

$$\lambda^2 + (\sigma+1)\lambda - \sigma(\rho-1) = 0$$

which yield

$$\lambda_{1,2} = \frac{-(\sigma+1) \pm \sqrt{(\sigma+1)^2 + 4\sigma(r-1)}}{2}.$$

If $r < 1$ the real parts of $\lambda_{1,2}$ are negative and the system is stable at the origin. If $r = 1$, one eigen value equals zero and the other is negative (the origin is a saddle point). If $r > 1$ at least one eigenvalue is positive. Although this does not suffice to ensure that the system is chaotic, it does suffice to establish instability of the origin. The Lorenz system exhibits a chaotic behaviour for certain values of the parameters σ, b and r. An illustrative plot is shown in Figure 1.2. An important characteristic to be noticed is that, since the system's trajectories converge to a strange attractor that may be strictly contained in a compact which depends on the size of the initial conditions.

Lü system

A modification of the Lorenz oscillator is the so-called Lü system. The third-order system is given by the following equations

$$\dot{x} = \beta x - yz + c \tag{1.8a}$$
$$\dot{y} = -ay + xz \tag{1.8b}$$
$$\dot{z} = -bz + xy \tag{1.8c}$$

where the constant parameters β, a and b are strictly positive and $c \in \mathbb{R}$. For the following values of the parameters: $a = -10$, $b = -4$, $c = 0$, $\beta = 2.8571$ the system exhibits a chaotic behaviour.

CHAPTER 2

Emergent Dynamics

1 Introduction

Of particular interest in the analysis of network-interconnected complex systems is to understand their ability to produce collective (synchronised) behaviour. As we explain in the Introduction, this depends on some key factors, such as:

- the dynamics of the individual units;
- the interconnection among the nodes;
- the network structure.

In what the nature of dynamics concerns, we must recognise that the analysis and control of networks of systems with hybrid dynamics have become a popular area of research during the last decade –see $e.g.$, [92, 48]. In this memoir we adopt the more common representation of the nodes' dynamics: via *ordinary* differential equations, represented by continuous-time models –$cf.$ [71, 72],

$$\dot{\boldsymbol{x}}_i = f_i(\boldsymbol{x}_i) + B\boldsymbol{u}_i, \qquad i \in \mathcal{I} := \{1, \dots, N\} \tag{2.1a}$$

$$\boldsymbol{y}_i = C\boldsymbol{x}_i, \tag{2.1b}$$

where $\boldsymbol{x}_i \in \mathbb{R}^n$, $\boldsymbol{u}_i \in \mathbb{R}^m$ and $\boldsymbol{y}_i \in \mathbb{R}^m$ denote the state, the input and the output of the ith unit, respectively. Usually, graph theory is employed to describe the topological (structural) properties of networks; a network of N nodes is defined by its $N \times N$ adjacency matrix $D = [d_{ij}]$ whose (i, j) element, denoted by d_{ij}, specifies an interconnection between the ith and jth nodes. See [73].

From a dynamical-systems viewpoint a general setting such as $e.g.$, in [12, 64], synchronisation may be qualitatively measured by equating a functional of the trajectories to zero and measuring the distance of the latter to the synchronisation manifold. Hence, the synchronisation problem may be recasted in terms of stability analysis of a manifold defined in function of the systems' states or, more generally, outputs. In the case of a network of identical nodes, $i.e.$, if $f_i = f_j$ for all $i, j \in \mathcal{I}$, synchronisation is often defined in terms of the (asymptotically) identical evolution of the units' motions hence, it may be formulated as a problem of the (asymptotic) stability of the synchronisation manifold

$$\mathcal{S} = \{\boldsymbol{x} \in \mathbb{R}^{nN} : \boldsymbol{x}_1 = \boldsymbol{x}_2 = \cdots = \boldsymbol{x}_N\} \tag{2.2}$$

in the space of $\boldsymbol{x} := [\boldsymbol{x}_1^\top, \; \cdots, \; \boldsymbol{x}_N^\top]^\top$. Such stability problem may be approached in a number of ways, $e.g.$, using tools developed for semi-passive, incrementally passive or incrementally input-output stable systems –see [71, 70, 44, 54, 77, 31]. If the manifold $\mathcal{S}$ is stabilised one says that the networked units are synchronised.

In general, the nodes' interconnections depend on the strength of the coupling and on the nodes' state variables or on functions of the latter, $i.e.$, outputs which define the coupling terms. The interaction is also determined by the form of coupling, $i.e.$, the way how the output of one node affects another; this can be linear, as it is fairly common to assume, but it may also be non linear, as in the well-known example of Kuramoto's oscillator model in which the interconnection is made via sinusoids –see [11, 24, 60, 28]. Other types of nonlinear nonlinear coupling appear in the realm of neural modelling –see [52, 87, 30].

In this memoir we consider a particular case of coupling which is known in the literature as *diffusive coupling*. We assume that all the units have inputs and outputs of the same dimension and that the coupling between the ith and jth units is defined as a weighted difference: $d_{ij}(\boldsymbol{y}_i - \boldsymbol{y}_j)$, where $\boldsymbol{y}_i$ and $\boldsymbol{y}_j$ are the outputs of the units i and j respectively, and $d_{ij} > 0$ is constant.

Thirdly, depending on whether the nodes are identical or not the network is called, respectively, *homogeneous* or *heterogeneous*. For a network of physically (structurally) similar units $e.g.$, predator-prey or neuronal network, heterogeneity may appear due to variations in the parameters that characterise these units. For example, the Hindmarsh-Rose model of neuronal dynamics is defined using seven parameters –see [37, 78, 88]. More generally, heterogeneous networks may be composed of systems with structurally different nodes.

The behaviour of networks of systems with non-identical models is more complex due to the fact that the synchronisation manifold $\mathcal{S}$ does not necessarily exist due, precisely, to the differences among the dynamics of the oscillators. Yet, it is well known from the literature on dynamical systems that such heterogeneous networks can exhibit some type of synchronisation and collective behaviour. One of the possible approaches to address this problem is to consider *practical synchronisation* that is, to admit that, asymptotically, the differences between the units' motions are bounded and become smaller for larger values of the interconnection gain γ, but they do not necessarily vanish. This is the approach that we pursue here.

We show that, for the purpose of analysis, the behaviour of the systems interconnected over the network via diffusive coupling may be studied via two separate properties: the stability of what we call the *emergent dynamics* and the synchronisation errors of each of the units' motions, relative to an averaged system, also called "mean-field" system. The emergent dynamics is a type of averaged model of the systems' dynamics regardless of the inputs while the mean-field oscillator's motion corresponds to the average of the units' motions and, as we shall see, its "steady-state" corresponds to the motion defined by the emergent dynamics. For instance, in the classical paradigm of consensus of a collection of integrators,

$$\dot{z}_i = u_i, \tag{2.3}$$

which is a particular case of our framework, the emergent dynamics is null while the mean field trajectory corresponds to a weighted average $z_m(t) = (1/N)\sum_{i=1}^{N} \omega_i z_i(t)$. For a balanced graph, we know that all units reach consensus and the steady-state value is an equilibrium point corresponding to the average of the initial conditions –see [74].

In our framework, the emergent dynamics possesses a stable attractor, in contrast to (the particular case of) an equilibrium point as is the case of (2.3). Then, we say that the network presents dynamic consensus if there exists an attractor $\mathcal{A}$, in the phase-space of the emergent state, such that the trajectories of all units are attracted to $\mathcal{A}$ asymptotically and remain close to it. This, however, is possible only for homogeneous networks. In the setting of heterogeneous networks, only $practical$ synchronisation is achievable in general that is, the trajectories of all units converge to a neighbourhood of the attractor of the emergent dynamics and remain close to this neighbourhood. The tools that we present in this chapter provide a formal description of practical synchronisation.

In the following sections we present our general framework for analysis of synchronisation of nonlinear systems over heterogeneous networks and we present a brief example of synchronisation of distinct chaotic systems. In Chapter 3 we present original and detailed material on the case-study of Stuart-Landau oscillators.

2 Networks of heterogeneous systems

2.1 System model

We consider a network composed of N heterogeneous diffusively coupled nonlinear dynamical systems in normal form:

$$\dot{\boldsymbol{y}}_i = f_i^1(\boldsymbol{y}_i, \boldsymbol{z}_i) + \boldsymbol{u}_i \tag{2.4a}$$

$$\dot{\boldsymbol{z}}_i = f_i^2(\boldsymbol{y}_i, \boldsymbol{z}_i). \tag{2.4b}$$

As it may be clear from the notation, each unit possesses one input $\boldsymbol{u}_i$, and one output $\boldsymbol{y}_i$ of the same dimension, $i.e.$, $\boldsymbol{u}_i,\ \boldsymbol{y}_i \in \mathbb{R}^m$. The state $\boldsymbol{z}_i$ corresponds to that of the ith agent's zero-dynamics –see [42, 43]. The functions $f_i^1 : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^m$, $f_i^2 : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^{n-m}$ are assumed to be locally Lipschitz.

It is convenient to remark that there is little loss of generality in considering systems in normal form, these are equivalent to systems of the form (2.1) under the assumption that the matrices $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{m\times n}$ satisfy a similarity condition for CB that is, if there exists U such that $U^{-1}CBU = \Lambda$ where Λ is diagonal positive –cf. [71, 72]. Of greater interest is that we assume that the systems are heterogeneous, that is, the functions f_i are, in general, different but of the same dimension (we describe the network model in detail farther below).

We also assume that the units possess certain physical properties reminiscent of energy dissipation and propagation. Notably, one of our main hypotheses is that the solutions are ultimately bounded; we recall the definition below.

Definition 2.1 (Ultimate boundedness) *The solutions of the system* $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$, $(t, \boldsymbol{x}_\circ) \mapsto \boldsymbol{x}$ *are said to be ultimately bounded if there exist positive constants* $\Delta_\circ$ *and* B_x *such that for every* $\Delta \in (0, \Delta_\circ)$, *there exists a positive constant* $T(\Delta)$ *such that, for all* $\boldsymbol{x}_\circ \in B_\Delta = \{\boldsymbol{x} \in \mathbb{R}^n : |\boldsymbol{x}| \leqslant \Delta\}$ *they satisfy*

$$|\boldsymbol{x}(t, \boldsymbol{x}_\circ)| \leqslant B_x \quad \text{for all} \quad t \geqslant T. \tag{2.5}$$

If this bound holds for any arbitrary large Δ *then the solutions are globally ultimately bounded.*

Ultimate boundedness is a reasonable assumption for the class of systems of interest in this memoir, such as oscillators. In a more general context, as we show in Section 4, ultimate boundedness holds, for instance, if the units are strictly semi-passive –$cf.$ [71, 72, 66, 53].

Our second main assumption is that the zero-dynamics is convergent, uniformly in the passive outputs, in a *practical* sense:

A1 *For any compact sets $\mathbb{B}_z \subset \mathbb{R}^{n-m}$, $\mathbb{B}_y \subset \mathbb{R}^m$, there exist continuously differentiable positive definite functions $V_{\circ i} : \mathbb{B}_z \to \mathbb{R}_+$ and constants $\bar{\alpha}_i$, $\beta_i > 0$, $i \in \mathcal{I}$, such that*

$$\nabla V_{\circ i}^\top(\boldsymbol{z}_1 - \boldsymbol{z}_2)\left[f_i^2(\boldsymbol{y}, \boldsymbol{z}_1) - f_i^2(\boldsymbol{y}, \boldsymbol{z}_2)\right] \leqslant -\bar{\alpha}_i|\boldsymbol{z}_1 - \boldsymbol{z}_2|^2 + \beta_i \tag{2.6}$$

for all $\boldsymbol{z}_1$, $\boldsymbol{z}_2 \in \mathbb{B}_z$ and $\boldsymbol{y} \in \mathbb{B}_y$.

2.2 Network model

We assume that the network units are connected via *diffusive coupling*, $i.e.$, for the i-th unit the coupling is given by

$$\boldsymbol{u}_i = -\sigma \sum_{j=1}^N d_{ij}(\boldsymbol{y}_i - \boldsymbol{y}_j) \tag{2.7}$$

where the scalar σ corresponds to the coupling gain between the units and the individual interconnections weights, d_{ij}, satisfy the property $d_{ij} = d_{ji}$. Assuming that the network graph is connected and undirected the interconnections amongst the nodes are completely defined by the adjacency matrix, $D = [d_{ij}]_{i,j\in\mathcal{I}}$, which is used to construct the corresponding Laplacian matrix,

$$L = \begin{bmatrix} \sum_{i=2}^N d_{1i} & -d_{12} & \dots & -d_{1N} \\ -d_{21} & \sum_{i=1,i\neq 2}^N d_{2i} & \dots & -d_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -d_{N1} & -d_{N2} & \dots & \sum_{i=1}^{N-1} d_{Ni}. \end{bmatrix}. \tag{2.8}$$

By construction, all row sums of L are equal to zero. Moreover, since L is symmetric and the network is connected it follows that all eigenvalues of the Laplacian matrix are real and, moreover, L has exactly one eigenvalue (say, λ_1) equal to zero, while others are positive, $i.e.$, $0 = \lambda_1 < \lambda_2 \leqslant \cdots \leqslant \lambda_N$.

Next, we introduce a compact notation that is convenient for our purposes of analysis. We introduce the vectors of outputs, inputs and states, respectively, by

$$\boldsymbol{y} = \begin{bmatrix} \boldsymbol{y}_1 \\ \vdots \\ \boldsymbol{y}_N \end{bmatrix} \in \mathbb{R}^{mN}, \quad \boldsymbol{u} = \begin{bmatrix} \boldsymbol{u}_1 \\ \vdots \\ \boldsymbol{u}_N \end{bmatrix} \in \mathbb{R}^{mN}, \quad \boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_1 \\ \vdots \\ \boldsymbol{x}_N \end{bmatrix} \in \mathbb{R}^{nN}, \quad \boldsymbol{x}_i = \begin{bmatrix} \boldsymbol{y}_i \\ \boldsymbol{z}_i \end{bmatrix} \in \mathbb{R}^n$$

and the function $F : \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ is defined as

$$F(\boldsymbol{x}) = \begin{bmatrix} F_1(\boldsymbol{x}_1) \\ \vdots \\ F_N(\boldsymbol{x}_N) \end{bmatrix}, \quad F_i(\boldsymbol{x}_i) = \begin{bmatrix} f_i^1(\boldsymbol{y}_i, \boldsymbol{z}_i) \\ f_i^2(\boldsymbol{y}_i, \boldsymbol{z}_i) \end{bmatrix}_{i\in\mathcal{I}}. \tag{2.9}$$

With this notation, the diffusive coupling inputs $\boldsymbol{u}_i$, defined in (2.7), can be re-written in the compact form

$$\boldsymbol{u} = -\sigma[L \otimes I_m]\boldsymbol{y},$$

where the symbol $\otimes$ stands for the right Kronecker product. Then, the network dynamics becomes

$$\dot{\boldsymbol{x}} = F(\boldsymbol{x}) - \sigma[L \otimes E_m]\boldsymbol{y} \tag{2.10a}$$

$$\boldsymbol{y} = [I_N \otimes E_m^\top]\boldsymbol{x}, \tag{2.10b}$$

where $E_m^\top = [I_m, 0_{m\times(n-m)}]$. The qualitative analysis of the solutions to the latter equations is our main subject of study; our analysis framework is described next.

2.3 Dynamic consensus and practical synchronisation

As we have explained, in the analysis of networks of identical nodes, $i.e.$, if $f_i = f_j$ for all $i, j \in \mathcal{I}$, synchronisation is often described in terms of the asymptotically identical evolution of the units, $i.e.$, $\boldsymbol{x}_i \to \boldsymbol{x}_j$ and, in the classical consensus paradigm we have $\boldsymbol{x}_i \to \boldsymbol{x}_j \to$ const. In more complex cases, as for instance in problem of formation tracking control, we may have that each unit follows a (possibly unique) reference trajectory, that is, $\boldsymbol{x}_i \to \boldsymbol{x}_j \to \boldsymbol{x}^*(t)$.

We generalise the consensus paradigm by introducing what we call *dynamic consensus*. We shall say that this property is achieved by the systems interconnected over a network if and only if their motions converge to one generated by what we shall call *emergent dynamics*. We stress that in the case that the Laplacian is symmetric, the emergent dynamics is naturally defined as the average of the units' drifts that is, the functions $f_s^1 : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^m$, $f_s^2 : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^{n-m}$ defined as

$$f_s^1(\boldsymbol{y}_e, \boldsymbol{z}_e) \;:=\; \frac{1}{N}\sum_{i=1}^{N} f_i^1(\boldsymbol{y}_e, \boldsymbol{z}_e), \tag{2.11a}$$

$$f_s^2(\boldsymbol{y}_e, \boldsymbol{z}_e) \;:=\; \frac{1}{N}\sum_{i=1}^{N} f_i^2(\boldsymbol{y}_e, \boldsymbol{z}_e) \tag{2.11b}$$

hence, the emergent dynamics may be written in the compact form

$$\dot{\boldsymbol{x}}_e = f_s(\boldsymbol{x}_e) \qquad \boldsymbol{x}_e = [\boldsymbol{y}_e^\top \, \boldsymbol{z}_e^\top]^\top, \quad f_s := [f_s^{1\top} \, f_s^{2\top}]^\top. \tag{2.12}$$

For the sake of comparison, in the classical (set-point) consensus paradigm, all systems achieving consensus converge to a common equilibrium *point* that is, $f_s \equiv 0$ and $\boldsymbol{x}_e$ is constant. In the case of formation tracking *control*, Equation (2.12) can be seen as the reference dynamics to the formation.

To better understand the behaviour of the systems interconnected over the network, relatively to that of the emergent dynamics, we introduce the *average* state (also called mean-field) and its corresponding dynamics. Let

$$\boldsymbol{x}_s = \frac{1}{N}\sum_{i=1}^{N} \boldsymbol{x}_i \tag{2.13}$$

which comprises an average output, $\boldsymbol{y}_s \in \mathbb{R}^m$, defined as $\boldsymbol{y}_s = E_m^\top \boldsymbol{x}_s$ and the state of the average zero dynamics, $\boldsymbol{z}_s \in \mathbb{R}^{n-m}$, that is, $\boldsymbol{x}_s = [\boldsymbol{y}_s^\top, \boldsymbol{z}_s^\top]^\top$. Now, by differentiating on both sides of (2.13) and after a direct computation in which we use (2.4), (2.7) and the fact that the sums of the elements of the Laplacian's rows equal to zero, *i.e.*,

$$\frac{1}{N}\sum_{i=1}^{N} -\sigma\big[d_{i1}(\boldsymbol{y}_i - \boldsymbol{y}_1) + \cdots + d_{iN}(\boldsymbol{y}_i - \boldsymbol{y}_N)\big] = 0,$$

we obtain

$$\dot{\boldsymbol{y}}_s = \frac{1}{N}\sum_{i=1}^{N} f_i^1(\boldsymbol{y}_i, \boldsymbol{z}_i), \tag{2.14a}$$

$$\dot{\boldsymbol{z}}_s = \frac{1}{N}\sum_{i=1}^{N} f_i^2(\boldsymbol{y}_i, \boldsymbol{z}_i). \tag{2.14b}$$

Next, in order to write the latter in terms of the average state $\boldsymbol{x}_s$, we use the functions f_s^1 and f_s^2 defined above so, from (2.14), we derive the *average dynamics*

$$\dot{\boldsymbol{y}}_s = f_s^1(\boldsymbol{y}_s, \boldsymbol{z}_s) + \frac{1}{N}\sum_{i=1}^{N}\big[f_i^1(\boldsymbol{y}_i, \boldsymbol{z}_i) - f_i^1(\boldsymbol{y}_s, \boldsymbol{z}_s)\big], \tag{2.15a}$$

$$\dot{\boldsymbol{z}}_s = f_s^2(\boldsymbol{y}_s, \boldsymbol{z}_s) + \frac{1}{N}\sum_{i=1}^{N}\big[f_i^2(\boldsymbol{y}_i, \boldsymbol{z}_i) - f_i^2(\boldsymbol{y}_s, \boldsymbol{z}_s)\big]. \tag{2.15b}$$

The latter equations may be regarded as composed of the nominal part

$$\dot{\boldsymbol{y}}_s = f_s^1(\boldsymbol{y}_s, \boldsymbol{z}_s) \tag{2.16a}$$

$$\dot{\boldsymbol{z}}_s = f_s^2(\boldsymbol{y}_s, \boldsymbol{z}_s) \tag{2.16b}$$

and the perturbation terms $\big[f_i^1(\boldsymbol{y}_i, \boldsymbol{z}_i) - f_i^1(\boldsymbol{y}_s, \boldsymbol{z}_s)\big]$ and $\big[f_i^2(\boldsymbol{y}_i, \boldsymbol{z}_i) - f_i^2(\boldsymbol{y}_s, \boldsymbol{z}_s)\big]$. The former corresponds exactly to (2.12), only re-written with another state variable. In the case that dynamic consensus is achieved (that is, in the case of complete synchronisation) and the graph is balanced and connected, we have $(\boldsymbol{y}_i, \boldsymbol{z}_i) \to (\boldsymbol{y}_s, \boldsymbol{z}_s)$. The latter is possible only for homogeneous connected and balanced networks. In the case of a heterogeneous network, asymptotic synchronisation cannot be achieved in general hence, $\boldsymbol{y}_i \not\to \boldsymbol{y}_s$ and, consequently, the terms $\big[f_i^1(\boldsymbol{y}_i, \boldsymbol{z}_i) - f_i^1(\boldsymbol{y}_s, \boldsymbol{z}_s)\big]$ and $\big[f_i^2(\boldsymbol{y}_i, \boldsymbol{z}_i) - f_i^2(\boldsymbol{y}_s, \boldsymbol{z}_s)\big]$ remain.

Thus, from a dynamical systems viewpoint, the average dynamics may be considered as a perturbed variant of the emergent dynamics. Consequently, it appears natural to study the problem of dynamic consensus, recasted in that of *robust* stability analysis, in a broad sense. On one hand, in contrast to the more-commonly studied case of state-synchronisation, we shall admit that synchronisation may be established with respect to part of the variables only, *i.e.*, with respect to the outputs $\boldsymbol{y}_i$. More precisely, for the former case, similarly to (2.2), we introduce the *state* synchronisation manifold

$$\mathcal{S}_x = \{\boldsymbol{x} \in \mathbb{R}^{nN} : \boldsymbol{x}_1 - \boldsymbol{x}_s = \boldsymbol{x}_2 - \boldsymbol{x}_s = \cdots = \boldsymbol{x}_N - \boldsymbol{x}_s = 0\} \tag{2.17}$$

and, for the study of *output* synchronisation, we analyse the stability of the manifold

$$\mathcal{S}_y = \{\boldsymbol{y} \in \mathbb{R}^{mN} : \boldsymbol{y}_1 - \boldsymbol{y}_s = \boldsymbol{y}_2 - \boldsymbol{y}_s = \cdots = \boldsymbol{y}_N - \boldsymbol{y}_s = 0\}. \tag{2.18}$$

In these terms, it seems fitting to reiterate that the manifold $\mathcal{S}_x$ can be stabilised *asymptotically* only in the case of homogeneous networks. For heterogeneous networks one may only aspire at establishing stability of the output or state synchronisation manifolds $\mathcal{S}_y$ or $\mathcal{S}_x$ in a *practical* sense. The following definition covers that of practical stability used in [85, 17], by considering a stability property with respect to sets.

Consider a parameterized system of differential equations

$$\dot{x} = f(x, \epsilon), \tag{2.19}$$

where $x \in R^n$, the function $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz and ϵ is a scalar parameter such that $\epsilon \in (0, \epsilon_\circ]$ with $\epsilon_\circ < \infty$. Given a closed set $\mathcal{A}$, we define the norm $|x|_\mathcal{A} := \inf_{y \in \mathcal{A}} |x - y|$.

Definition 2.2 *For the system (2.19), we say that the closed set $\mathcal{A} \subset \mathbb{R}^n$ is practically uniformly asymptotically stable if there exists a closed set $\mathcal{D}$ such that $\mathcal{A} \subset \mathcal{D} \subset \mathbb{R}^n$ and:*

(1) the system is forward complete for all $x_\circ \in \mathcal{D}$;

(2) for any given $\delta > 0$ and $R > 0$, there exist $\epsilon^* \in (0, \epsilon_\circ]$ and a class $\mathcal{KL}$ function $\beta_{\delta R}$ such that, for all $\epsilon \in (0, \epsilon^*]$ and all $x_\circ \in \mathcal{D}$ such that $|x_\circ|_\mathcal{A} \leqslant R$, we have

$$|x(t, x_\circ, \epsilon)|_\mathcal{A} \leqslant \delta + \beta_{\delta R}\big(|x_\circ|_\mathcal{A}, t\big).$$

Remark 1 *Similarly to the definition of uniform global asymptotic stability of a set, the previous definition includes three properties: uniform boundedness of the solutions with respect to the set, uniform stability of the set and uniform practical convergence to the set.*

The following statement, which establishes practical asymptotic stability of sets, may be deduced from [23].

Proposition 1 *Consider the system $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$, where $\boldsymbol{x} \in \mathbb{R}^n$ and f is continuous, locally Lipschitz. Assume that the system is forward complete, there exists a closed set $\mathcal{A} \subset \mathbb{R}^n$ and a C^1 function $V : \mathbb{R}^n \to \mathbb{R}_+$ as well as functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\alpha_3 \in \mathcal{K}$ and a constant $c > 0$, such that, for all $\boldsymbol{x} \in \mathbb{R}^n$,*

$$\alpha_1(|\boldsymbol{x}|_\mathcal{A}) \leqslant V(\boldsymbol{x}) \leqslant \alpha_2(|\boldsymbol{x}|_\mathcal{A})$$
$$\dot{V} \leqslant -\alpha_3(|\boldsymbol{x}|_\mathcal{A}) + c.$$

Then for any $R, \epsilon > 0$ there exists a constant $T = T(R, \epsilon)$ such that for all $t \geqslant T$ and all $\boldsymbol{x}_\circ$ such that $|\boldsymbol{x}_\circ|_\mathcal{A} \leqslant R$

$$|\boldsymbol{x}(t, \boldsymbol{x}_\circ)|_\mathcal{A} \leqslant r + \epsilon,$$

where $r = \alpha_1^{-1} \circ \alpha_2 \circ \alpha_3^{-1}(c)$.

3 Network dynamics

In the previous section we motivate, albeit intuitively, the study of dynamic consensus and practical synchronisation as a stability problem of the attractor of the emergent dynamics as well as of the synchronisation manifold. In this section, we render this argument formal by showing that the networked dynamical systems model (2.10) is equivalent, up to a coordinate transformation, to a set of equations composed of the average system dynamics (2.15) with average state $\boldsymbol{x}_s$ and a synchronisation errors equation with state $\boldsymbol{e} = [\boldsymbol{e}_1^\top \, \dots \, \boldsymbol{e}_N^\top]^\top$ where $\boldsymbol{e}_i = \boldsymbol{x}_i - \boldsymbol{x}_s$ for all $i \in \mathcal{I}$. It is clear that $\boldsymbol{x} \in \mathcal{S}_x$ if and only if $\boldsymbol{e} = 0$ hence, the general synchronisation problem is recasted in the study of stability of the dynamics of $\boldsymbol{e}$ and $\boldsymbol{x}_s$.

3.1 New coordinates

We formally justify that the choice of coordinates $\boldsymbol{x}_s$ and $\boldsymbol{e}$ completely and appropriately describe the networked systems' behaviours.

If the graph of the network is undirected and connected, the Laplacian matrix $L = L^\top$ has a single zero eigenvalue $\lambda_1 = 0$ and its corresponding right and left eigenvectors v_{r1}, v_{l1} coincide with $v = \frac{1}{\sqrt{N}}\mathbf{1}$ where $\mathbf{1} \in \mathbb{R}^N$ denotes the vector $[1\ 1\ \cdots\ 1]^\top$. Moreover, since L is symmetric and non-negative definite, there exists (see [10, Chapter 4, Theorems 2 and 3]) an orthogonal matrix U ($i.e.$, $U^{-1} = U^\top$) such that $L = U\Lambda U^\top$ with $\Lambda = \text{diag}\{[0\,\lambda_2 \cdots \lambda_N]\}$, where $\lambda_i > 0$ for all $i \in [2, N]$, are the eigenvalues of L. Furthermore, the i-th column of U corresponds to an eigenvector of L related to the i-th eigenvalue, λ_i. Therefore, recognising v, as the first eigenvector, we decompose the matrix U as:

$$U = \left[\frac{1}{\sqrt{N}}\mathbf{1}\ U_1\right], \tag{2.20}$$

where $U_1 \in \mathbb{R}^{N \times N-1}$ is a matrix composed of $N-1$ eigenvectors of L related to $\lambda_2, \dots, \lambda_N$ and, since the eigenvectors of a real symmetric matrix are orthogonal, we have

$$\frac{1}{\sqrt{N}}\mathbf{1}^\top U_1 = 0, \qquad U_1^\top U_1 = I_{N-1}.$$

Based on the latter observations we introduce the coordinate transformation

$$\bar{\boldsymbol{x}} = \mathcal{U}^\top \boldsymbol{x}, \tag{2.21}$$

where the block diagonal matrix $\mathcal{U} \in \mathbb{R}^{nN \times nN}$ is defined as

$$\mathcal{U} = U \otimes I_n \tag{2.22}$$

which, in view of (2.21), is also orthogonal. Then, we use (2.20) to partition the new coordinates $\bar{\boldsymbol{x}}$, $i.e.$,

$$\bar{\boldsymbol{x}} = \begin{bmatrix} \bar{\boldsymbol{x}}_1 \\ \bar{\boldsymbol{x}}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{N}}\mathbf{1}_N^\top \otimes I_n \\ U_1^\top \otimes I_n \end{bmatrix} \boldsymbol{x}.$$

The coordinates $\bar{\boldsymbol{x}}_1$ and $\bar{\boldsymbol{x}}_2$ thus obtained are equivalent to the average $\boldsymbol{x}_s$ and the synchronisation errors $\boldsymbol{e}$, respectively. Indeed, observing that the state of the average unit, defined in (2.13), may be re-written in the compact form

$$\boldsymbol{x}_s = \frac{1}{N}(\mathbf{1}^\top \otimes I_n)\boldsymbol{x}, \tag{2.23}$$

we have $\bar{\boldsymbol{x}}_1 = \sqrt{N}\boldsymbol{x}_s$ while $\bar{\boldsymbol{x}}_2 = 0$ if and only if $\boldsymbol{e} = 0$. To see the latter, let $\mathcal{U}_1 = U_1 \otimes I_n$ then, using the expression

$$(A \otimes B)(C \otimes D) = AC \otimes BD, \tag{2.24}$$

we obtain

$$\mathcal{U}_1\mathcal{U}_1^\top = (U_1U_1^\top) \otimes I_n$$

and, observing that

$$U_1U_1^\top = I_N - \frac{1}{N}\mathbf{1}\mathbf{1}^\top, \tag{2.25}$$

we get

$$\mathcal{U}_1\mathcal{U}_1^\top = \left(I_N - \frac{1}{N}\mathbf{1}\mathbf{1}^\top\right) \otimes I_n. \tag{2.26}$$

So, multiplying $\bar{\boldsymbol{x}}_2 = \mathcal{U}_1^\top \boldsymbol{x}$ by $\mathcal{U}_1$ and using (2.26), we obtain

$$\begin{aligned} \mathcal{U}_1\bar{\boldsymbol{x}}_2 &= \left[\left(I_N - \frac{1}{N}\mathbf{1}\mathbf{1}^\top\right) \otimes I_n\right]\boldsymbol{x} \\ &= \boldsymbol{x} - \frac{1}{N}\big(\mathbf{1}\mathbf{1}^\top \otimes I_n\big)\boldsymbol{x} \end{aligned}$$

which, in view of (2.24), is equivalent to

$$\begin{aligned} \mathcal{U}_1\bar{\boldsymbol{x}}_2 &= \boldsymbol{x} - \frac{1}{N}\big(\mathbf{1} \otimes I_n\big)\big(\mathbf{1}^\top \otimes I_n\big)\boldsymbol{x} \\ &= \boldsymbol{x} - \big(\mathbf{1} \otimes I_n\big)\boldsymbol{x}_s = \boldsymbol{e}. \end{aligned}$$

Thus, since $\mathcal{U}_1$ has column rank equal to $(N-1)n$, which corresponds to the dimension of $\bar{\boldsymbol{x}}_2$, we see that $\bar{\boldsymbol{x}}_2$ is equal to zero if and only if so is $\boldsymbol{e}$.

Thus, the states $\boldsymbol{x}_s$ and $\boldsymbol{e}$ are *intrinsic* to the network and not the product of an artifice with purely theoretical motivations. We proceed to derive the differential equations in terms of the average state $\boldsymbol{x}_s$ and the synchronisation errors $\boldsymbol{e}$.

3.2 Dynamics of the average unit

Using the network dynamics equations (2.10a), as well as (2.23), we obtain

$$\dot{\boldsymbol{x}}_s = \frac{1}{N}(\mathbf{1}^\top \otimes I_n)F(\boldsymbol{x}) - \frac{1}{N}\sigma(\mathbf{1}^\top \otimes I_n)[L \otimes E_m]\boldsymbol{y}. \tag{2.27}$$

Now, using the property of the Kronecker product, (2.24), and in view of the identity $\mathbf{1}^\top L = 0$, we obtain

$$(\mathbf{1}^\top \otimes I_n)(L \otimes E_m) = (\mathbf{1}^\top L) \otimes (I_nE_m) = 0. \tag{2.28}$$

This reveals the important fact that the average dynamics, $i.e.$, the right-hand side of (2.27), is independent of the interconnections gain σ, even though the solutions $\boldsymbol{x}_s(t)$ are, certainly, affected by the synchronisation errors hence, by the coupling strength.

Now, using (2.9) and defining

$$f_s(\boldsymbol{x}_s) := \frac{1}{N}\sum_{i=1}^{N} F_i(\boldsymbol{x}_s) \tag{2.29}$$

we obtain

$$\dot{\boldsymbol{x}}_s = f_s(\boldsymbol{x}_s) + \frac{1}{N}\sum_{i=1}^{N}\Big[F_i(\boldsymbol{x}_i) - F_i(\boldsymbol{x}_s)\Big]$$

therefore, defining

$$G_s(\boldsymbol{e}, \boldsymbol{x}_s) := \frac{1}{N}\sum_{i=1}^{N}\Big[F_i(\boldsymbol{e}_i + \boldsymbol{x}_s) - F_i(\boldsymbol{x}_s)\Big], \tag{2.30}$$

we see that we may write the average dynamics in the compact form,

$$\dot{\boldsymbol{x}}_s = f_s(\boldsymbol{x}_s) + G_s(\boldsymbol{e}, \boldsymbol{x}_s). \tag{2.31}$$

Furthermore, since the functions F_i, with $i \in \mathcal{I}$, are locally Lipschitz so is the function G_s and, moreover, there exists a continuous, positive, non-decreasing function $k : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, such that

$$|G_s(\boldsymbol{e}, \boldsymbol{x}_s)| \leqslant k\big(\,|\boldsymbol{e}|\,, |\boldsymbol{x}_s|\,\big)\,|\boldsymbol{e}|\,. \tag{2.32}$$

In summary, the average dynamics is described by the equations (2.31), which may be regarded as the nominal system (2.12), which corresponds to the emergent dynamics, perturbed by the synchronisation error of the network.

3.3 Dynamics of the synchronisation errors

To study the effect of the synchronisation errors, $\boldsymbol{e}(t)$, on the emergent dynamics, we start by introducing the vectors

$$F_s(\boldsymbol{x}_s) := \big[F_1(\boldsymbol{x}_s)^\top \quad \cdots \quad F_N(\boldsymbol{x}_s)^\top\big]^\top \tag{2.33}$$

$$\tilde{F}(\boldsymbol{e}, \boldsymbol{x}_s) := \begin{bmatrix} F_1(\boldsymbol{x}_1) - F_1(\boldsymbol{x}_s) \\ \vdots \\ F_N(\boldsymbol{x}_N) - F_N(\boldsymbol{x}_s) \end{bmatrix} = \begin{bmatrix} F_1(\boldsymbol{e}_1 + \boldsymbol{x}_s) - F_1(\boldsymbol{x}_s) \\ \vdots \\ F_N(\boldsymbol{e}_N + \boldsymbol{x}_s) - F_N(\boldsymbol{x}_s) \end{bmatrix} \tag{2.34}$$

$i.e.$, $\tilde{F}(\boldsymbol{e}, \boldsymbol{x}_s) = F(\boldsymbol{x}) - F_s(\boldsymbol{x}_s)$. Then, differentiating on both sides of

$$\boldsymbol{e} = \boldsymbol{x} - (\mathbf{1} \otimes I_n)\boldsymbol{x}_s \tag{2.35}$$

and using (2.10a) and (2.31), we obtain

$$\begin{aligned}
\dot{\boldsymbol{e}} &= -\sigma[L\otimes E_m]\boldsymbol{y} + F(\boldsymbol{x}) - (\mathbf{1}\otimes I_n)\left[f_s(\boldsymbol{x}_s) + G_s(\boldsymbol{e},\boldsymbol{x}_s)\right] \\
&= -\sigma[L\otimes E_m]\boldsymbol{y} + [F(\boldsymbol{x}) - F_s(\boldsymbol{x}_s)] + F_s(\boldsymbol{x}_s) - (\mathbf{1}\otimes I_n)\left[f_s(\boldsymbol{x}_s) + G_s(\boldsymbol{e},\boldsymbol{x}_s)\right] \\
&= -\sigma[L\otimes E_m]\boldsymbol{y} + \big[F_s(\boldsymbol{x}_s) - (\mathbf{1}\otimes I_n)f_s(\boldsymbol{x}_s)\big] + \big[\tilde{F}(\boldsymbol{e},\boldsymbol{x}_s) - (\mathbf{1}\otimes I_n)G_s(\boldsymbol{e},\boldsymbol{x}_s)\big]. \qquad (2.36)
\end{aligned}$$

Next, let us introduce the output synchronisation errors $\boldsymbol{e}_{yi} = \boldsymbol{y}_i - \boldsymbol{y}_s$, $\boldsymbol{e}_y = [\boldsymbol{e}_{y1}^\top, \cdots, \boldsymbol{e}_{yN}^\top]^\top$, which may also be written as

$$\boldsymbol{e}_y = \boldsymbol{y} - \mathbf{1}\otimes\boldsymbol{y}_s, \qquad (2.37)$$

and let us consider the first term and the two groups of bracketed terms on the right-hand side of (2.36), separately. For the term $\big(L\otimes E_m\big)\boldsymbol{y}$ we observe, from (2.37), that

$$[L\otimes E_m]\,\boldsymbol{y} = [L\otimes E_m]\left[\boldsymbol{e}_y + \mathbf{1}\otimes\boldsymbol{y}_s\right]$$

and we use (2.24) and the fact that $L\mathbf{1} = 0$ to obtain

$$[L\otimes E_m]\,\boldsymbol{y} = [L\otimes E_m]\,\boldsymbol{e}_y.$$

Secondly, concerning the first bracket on the right-hand side of (2.36) we observe that, in view of (2.29) and (2.33),

$$f_s(\boldsymbol{x}_s) = \frac{1}{N}(\mathbf{1}^\top\otimes I_n)F_s(\boldsymbol{x}_s)$$

therefore,

$$F_s(\boldsymbol{x}_s) - (\mathbf{1}\otimes I_n)f_s(\boldsymbol{x}_s) = F_s(\boldsymbol{x}_s) - \frac{1}{N}(\mathbf{1}\otimes I_n)(\mathbf{1}^\top\otimes I_n)F_s(\boldsymbol{x}_s).$$

Then, using (2.24) we see that

$$\frac{1}{N}(\mathbf{1}\otimes I_n)(\mathbf{1}^\top\otimes I_n) = \frac{1}{N}(\mathbf{1}\mathbf{1}^\top)\otimes I_n \qquad (2.38)$$

so, introducing

$$P = I_{nN} - \frac{1}{N}(\mathbf{1}\mathbf{1}^\top)\otimes I_n,$$

we obtain

$$F_s(\boldsymbol{x}_s) - (\mathbf{1}\otimes I_n)f_s(\boldsymbol{x}_s) = PF_s(\boldsymbol{x}_s). \qquad (2.39)$$

Finally, concerning the term $\tilde{F}(\boldsymbol{e},\boldsymbol{x}_s) - (\mathbf{1}\otimes I_n)G_s(\boldsymbol{e},\boldsymbol{x}_s)$ on the right-hand side of (2.36), we see that, by definition, $G(\boldsymbol{e},\boldsymbol{x}_s) = \frac{1}{N}\big(\mathbf{1}^\top\otimes I_n\big)\tilde{F}(\boldsymbol{e},\boldsymbol{x}_s)$ hence, from (2.38), we obtain

$$(\mathbf{1}\otimes I_n)G_s(\boldsymbol{e},\boldsymbol{x}_s) = \frac{1}{N}\big[(\mathbf{1}\mathbf{1}^\top)\otimes I_n\big]\tilde{F}(\boldsymbol{e},\boldsymbol{x}_s)$$

and

$$\tilde{F}(e,\boldsymbol{x}_s) - (\mathbf{1}\otimes I_n)G_s(\boldsymbol{e},\boldsymbol{x}_s) = \left(I_{nN} - \frac{1}{N}(\mathbf{1}\mathbf{1}^\top)\otimes I_n\right)\tilde{F}(\boldsymbol{e},\boldsymbol{x}_s) = P\tilde{F}(\boldsymbol{e},\boldsymbol{x}_s). \qquad (2.40)$$

Using (2.39) and (2.40) in (2.36) we see that the latter may be expressed as

$$\dot{\boldsymbol{e}} = -\sigma\big[L \otimes E_m\big]\boldsymbol{e}_y + P\big[\tilde{F}(\boldsymbol{e}, \boldsymbol{x}_s) + F_s(\boldsymbol{x}_s)\big].$$

The utility of this equation is that it clearly exhibits three terms: a term linear in the output $\boldsymbol{e}_y$ which reflects the synchronisation effect of diffusive coupling between the nodes, the term $P\tilde{F}(\boldsymbol{e}, \boldsymbol{x}_s)$ which vanishes with the synchronisation errors, $i.e.$, if $\boldsymbol{e} = 0$, and the term

$$PF_s(\boldsymbol{x}_s) = \begin{bmatrix} F_1(\boldsymbol{x}_s) - \frac{1}{N}\sum_{i=1}^{N} F_i(\boldsymbol{x}_s) \\ \vdots \\ F_N(\boldsymbol{x}_s) - \frac{1}{N}\sum_{i=1}^{N} F_i(\boldsymbol{x}_s) \end{bmatrix} = \begin{bmatrix} F_1(\boldsymbol{x}_s) - f_s(\boldsymbol{x}_s) \\ \vdots \\ F_N(\boldsymbol{x}_s) - f_s(\boldsymbol{x}_s) \end{bmatrix}$$

which represents the variation between the dynamics of the individual units and the average unit. This term equals to zero when the nominal dynamics, f_i in (2.1a), of all the units are identical that is, in the case of a homogeneous network.

4 Stability analysis

In this section we present our main statements on stability of the networked systems model (2.10). For the purpose of analysis we use the equations previously developed, in the coordinates $\boldsymbol{e}$ and $\boldsymbol{x}_s$, which we recall here for convenience:

$$\dot{\boldsymbol{x}}_s = f_s(\boldsymbol{x}_s) + G_s(\boldsymbol{e}, \boldsymbol{x}_s), \tag{2.41a}$$
$$\dot{\boldsymbol{e}} = -\sigma\big[L \otimes E_m\big]\boldsymbol{e}_y + P\big[\tilde{F}(\boldsymbol{e}, \boldsymbol{x}_s) + F_s(\boldsymbol{x}_s)\big]. \tag{2.41b}$$

We study the stability with respect to a compact attractor which is proper to the emergent dynamics and we establish conditions under which the average of the trajectories of the interconnected units remains close to this attractor. More precisely, firstly, we establish a stability result for the subsystem (2.41b) and, secondly, we analyse the subsystem (2.41a).

4.1 Network practical synchronisation under diffusive coupling

We formulate conditions that ensure practical global asymptotic stability of the sets $\mathcal{S}_x$ and $\mathcal{S}_y$ –see (2.17), (2.18). This implies practical state and output synchronisation of the network, respectively. Furthermore, we show that the upper bound on the state synchronisation error depends on the mismatches between the dynamics of the individual units of the network.

Theorem 1 (Output synchronization) *Let the solutions of the system (2.10) be globally ultimately bounded. Then, the set $\mathcal{S}_y$ is practically uniformly globally asymptotically stable with $\epsilon = 1/\sigma$. If, moreover, Assumption* **A1** *holds, then there exists a function $\beta \in \mathcal{K}_\infty$ such that for any $\epsilon \geqslant 0$ and $R > 0$ there exist $T^* > 0$ and $\sigma^* > 0$ such that the solutions of (2.41b) with $\sigma = \sigma^*$ satisfy*

$$|\boldsymbol{e}(t, \boldsymbol{x}_\circ)| \leqslant \beta(\Delta_f) + \epsilon, \qquad \forall\, t \geqslant T^*,\ \boldsymbol{x}_\circ \in B_R := \{\boldsymbol{x}_\circ : |\boldsymbol{x}_\circ| \leqslant R\} \tag{2.42}$$

where

$$\Delta_f = \max_{|\boldsymbol{x}|\leqslant B_x} \max_{k,i\in N} \Big\{ \big| f_k^2(\boldsymbol{x}_k) - f_i^2(\boldsymbol{x}_k) \big| \Big\}. \tag{2.43}$$

The proof of the theorem is provided farther below. Roughly speaking, the first statement (synchronisation) follows from two properties of the networked system –namely, negative definiteness of the second smallest eigenvalue of the Laplacian metric L and global ultimate boundedness. As we establish next, in a statement reminiscent of [70, Corollary 1], global ultimate boundedness holds, $e.g.$, under the following assumption.

A2 *All the units (2.4) are strictly semi-passive with respect to the input u_i and output y_i with continuously differentiable and radially unbounded storage functions $V_i : \mathbb{R}^n \to \mathbb{R}_+$, where $i \in \mathcal{I}$. That is, there exist positive definite and radially unbounded storage functions V_i, positive constants ρ_i, continuous functions H_i and positive continuous functions ϱ_i such that*

$$\dot{V}_i(\boldsymbol{x}_i) \leqslant \boldsymbol{y}_i^\top \boldsymbol{u}_i - H_i(\boldsymbol{x}_i) \tag{2.44}$$

and $H_i(\boldsymbol{x}_i) \geqslant \varrho_i(|\boldsymbol{x}_i|)$ for all $|\boldsymbol{x}_i| \geqslant \rho_i$.

Proposition 2 *Consider a network of N diffusively coupled units (2.10). Let the graph of interconnections be undirected and connected and assume that all the units of the network are strictly semi-passive ($i.e.$, Assumption* **A2** *holds). Then, the solutions of the system (2.10) are ultimately bounded.*

Proof. We proceed as in the proof of [70, Lemma 1] and [80, Proposition 2.1]. Let Assumption **A2** generate positive definite storage functions V_i, as well as functions ϱ_i, H_i and constants ρ_i, defined as above and let

$$V_\Sigma(\boldsymbol{x}) := \sum_{i=1}^N V_i(\boldsymbol{x}_i).$$

Then, taking the derivative of $V_\Sigma(\boldsymbol{x})$ along trajectories of the system (2.10), we obtain

$$\begin{aligned} \dot{V}_\Sigma(\boldsymbol{x}) &\leqslant -\sigma \boldsymbol{y}^\top \big[L \otimes I_m\big] \boldsymbol{y} - \sum_{i=1}^N H_i(\boldsymbol{x}_i) \\ &\leqslant -\sum_{i=1}^N H_i(\boldsymbol{x}_i), \end{aligned} \tag{2.45}$$

where for the last inequality we used the fact that Laplacian matrix is semi-positive definite. Next, let $\bar{\rho} = \max_{1\leqslant i\leqslant N}\{\rho_i\}$ and consider the function $\bar{\varrho} : [\bar{\rho}, +\infty) \to \mathbb{R}_{\geqslant 0}$ as $\bar{\varrho}(s) = \min_{1\leqslant i\leqslant N}\{\varrho_i(s)\}$. Note that $\bar{\varrho}$ is continuous and $\bar{\varrho}(s)$ positive for all $s \geqslant \bar{\rho}$. Furthermore, for any $|\boldsymbol{x}| \geqslant N\bar{\rho}$ there exists $k \in \mathcal{I}$ such that $|\boldsymbol{x}_k| \geqslant \frac{1}{N}|\boldsymbol{x}| \geqslant \bar{\rho}$. Therefore, for all $|\boldsymbol{x}| \geqslant N\bar{\rho}$,

$$\sum_{i=1}^N H_i(\boldsymbol{x}_i) \geqslant H_k(\boldsymbol{x}_k) \geqslant \bar{\varrho}(|\boldsymbol{x}_k|) \geqslant \bar{\varrho}\Big(\frac{1}{N}|\boldsymbol{x}|\Big).$$

Using the last bound in (2.45) we obtain, for all $|\boldsymbol{x}| \geqslant N\bar{\rho}$,

$$\dot{V}_\Sigma(\boldsymbol{x}) \leqslant -\bar{\varrho}\Big(\frac{1}{N}|\boldsymbol{x}|\Big).$$

Hence, invoking [91, Theorem 10.4] we conclude that the solutions of the system (2.10) are ultimately bounded. ■

Some interesting corollaries, on state synchronization, follow from Theorem 1. For instance, if the interconnections among the network units depend on the whole state, that is, if $\boldsymbol{y} = \boldsymbol{x}$.

Corollary 1 *Consider the system (2.10). Let Assumptions* **A1** *and* **A2** *be satisfied and let $\boldsymbol{y} = \boldsymbol{x}$. Then, the system is forward complete and the set $\mathcal{S}_x$ is practically uniformly globally asymptotically stable with $\epsilon = 1/\sigma$.*

Proof. By Assumption **A2**, Proposition 2 holds so the system is forward complete and, moreover, the solutions are globally ultimately bounded. Therefore, the statement follows straightforward by invoking Theorem 1 with the output $\boldsymbol{y} = \boldsymbol{x}$ and $\boldsymbol{e}_y = \boldsymbol{x} - \mathbf{1} \otimes \boldsymbol{x}_s = \boldsymbol{e}$. ■

The constant Δ_f represents the maximal possible mismatch between the dynamics of any individual unit and that of the averaged unit, on a ball of radius B_x. The more heterogeneous is the network, the bigger is the constant Δ_f. Conversely, in the case that all the zero dynamics of the units are identical, we have $\Delta_f = 0$.

Corollary 2 *Consider the system (2.10) under Assumptions* **A1** *and* **A2**. *Assume that the functions f_i^2, which define zero dynamics of the network units, are all identical i.e., $f_i^2(x) = f_j^2(x)$ for all $i, j \in \mathcal{I}$ and all $x \in \mathbb{R}^n$. Then the set $\mathcal{S}_x$ is practically uniformly globally asymptotically stable with $\epsilon = 1/\sigma$.*

Proof of Theorem 1

Proof of the first statement. From ultimate boundedness there exist constants $B_x > 0$ and $T = T(R) > 0$ such that (2.5) holds. Moreover, as showed in the previous section, the equations (2.10) are equivalent to (2.41). We proceed to analyse the latter. For the system (2.41b) let us introduce the Lyapunov function candidate $V_y : \mathbb{R}^{mN} \to \mathbb{R}_+$ defined as $V_y(\boldsymbol{e}_y) = \frac{1}{2}\boldsymbol{e}_y^\top \boldsymbol{e}_y$, where

$$\boldsymbol{e}_y = \left[I_N \otimes E_m^\top\right]\boldsymbol{e} = \Big(I_N \otimes \left[I_m, 0_{m\times(n-m)}\right]\Big)\boldsymbol{e}$$

–cf. Equation (2.37). Differentiating $V_y(\boldsymbol{e}_y)$ along trajectories of (2.41b) we obtain

$$\begin{aligned}\dot{V}_y(\boldsymbol{e}_y) \;&=\; -\sigma\, \boldsymbol{e}_y^\top (I_N \otimes E_m^\top)(L \otimes E_m)\boldsymbol{e}_y + \boldsymbol{e}_y^\top (I_N \otimes E_m^\top) P\tilde{F}(\boldsymbol{e}, \boldsymbol{x}_s) \\ &\quad + \boldsymbol{e}_y^\top (I_N \otimes E_m^\top) P F_s(\boldsymbol{x}_s).\end{aligned}$$

Then, using the identities $(I_N \otimes E_m^\top)(L \otimes E_m) = L \otimes (E_m^\top E_m) = L \otimes I_m$ and the triangle inequality we obtain

$$\begin{aligned}\dot{V}_y(\boldsymbol{e}_y) \;&\leqslant\; -\sigma \boldsymbol{e}_y^\top (L \otimes I_m)\boldsymbol{e}_y + \Big[\big|P\tilde{F}(\boldsymbol{e}, \boldsymbol{x}_s)\big| + \big|PF_s(\boldsymbol{x}_s)\big|\Big]\,|\boldsymbol{e}_y| \\ &\leqslant\; -\sigma \lambda_2(L)\,|\boldsymbol{e}_y|^2 + \Big[\big|P\tilde{F}(\boldsymbol{e}, \boldsymbol{x}_s)\big| + \big|PF_s(\boldsymbol{x}_s)\big|\Big]\,|\boldsymbol{e}_y|\,.\end{aligned}$$

Now, since the solutions are ultimately bounded and (2.5) holds, there exists a constant C' such that $|\boldsymbol{x}_s(t)| \leqslant C'$. On the other hand, for all $|\boldsymbol{x}_s| \leqslant C'$ there also exist constants $C_1, C_2 > 0$ such that

$$\left|P\tilde{F}(\boldsymbol{e},\boldsymbol{x}_s)\right| \leqslant C_1 \left|\boldsymbol{e}_y\right|$$
$$\left|PF_s(\boldsymbol{x}_s)\right| \leqslant C_2$$

therefore,

$$\begin{aligned}\dot{V}_y(\boldsymbol{e}_y(t)) &\leqslant -\sigma\left[\lambda_2(L) - C_1\right]\left|\boldsymbol{e}_y(t)\right|^2 + C_2\left|\boldsymbol{e}_y(t)\right| \\ &\leqslant -\sigma\left[\lambda_2(L) - C_1 - \frac{C_2}{2}\right]\left|\boldsymbol{e}_y(t)\right|^2 + \frac{C_2}{2}.\end{aligned}$$

Notice that the constants C_1 and C_2 depend only on the functions $\tilde{F}$, F and on the constant B_x and are independent of σ. Now, denoting $C = C_1 + \frac{C_2}{2}$ we obtain that for all $\sigma \geqslant \sigma^* = 2C/\lambda_2(L)$,

$$\dot{V}_y(\boldsymbol{e}_y(t)) \leqslant -\frac{1}{2}\sigma\lambda_2(L)\left|\boldsymbol{e}_y(t)\right|^2 + \frac{C_2}{2}. \tag{2.46}$$

Next, applying Proposition 1 with $V(\boldsymbol{e}) = V_y(\boldsymbol{e}_y)$, $\mathcal{Z} := \mathcal{S}_y$, $\alpha_1(s) = \alpha_2(s) = \frac{1}{2}s^2$ and $\alpha_3(s) = \frac{1}{2}\sigma\lambda_2(L)s^2$, we obtain that, for any $\epsilon > 0$, there exists a $T = T(\epsilon) > 0$ such that

$$\left|\boldsymbol{e}_y(t,\boldsymbol{x}_\circ)\right| \leqslant r + \epsilon, \quad \forall t \geqslant T,$$

where $r = \sqrt{C_2/\sigma\lambda_2(L)}$. Since r is inversely proportional to $\sqrt{\sigma}$, it follows that $r \to 0$ as $\sigma \to +\infty$. Therefore the set S_y is practically uniformly globally asymptotically stable.

Proof of the second statement. Following Assumption **A1** we introduce the continuously differentiable Lyapunov function candidate $V_z : \mathbb{R}^{(n-m)N} \to \mathbb{R}_+$ defined as

$$V_z(\boldsymbol{z}) = \sum_{k=1}^{N}\sum_{i=1}^{N} V_{\circ k}(\boldsymbol{z}_k - \boldsymbol{z}_i) \tag{2.47}$$

and we make the following technical statement.

Claim 1 *There exist functions $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$ such that V_z satisfies the bounds*

$$\gamma_1(|\boldsymbol{e}_z|) \leqslant V_z(\boldsymbol{z}) \leqslant \gamma_2(|\boldsymbol{e}_z|), \tag{2.48}$$

where $\boldsymbol{e}_z := \left[I_N \otimes E_{n-m}^\top\right]\boldsymbol{e} = \boldsymbol{z} - \mathbf{1}_N \otimes \boldsymbol{z}_s$.

Proof of Claim 1. From Assumption **A1** there exist class $\mathcal{K}_\infty$ functions α_k^1, α_k^2 and sets $\mathbb{B}_z^k \subset \mathbb{R}^{n-m}$, with $k \in \mathcal{I}$, such that the functions $V_{\circ k}$ satisfy, for all $z_k \in \mathbb{B}_z^k$, the bounds

$$\alpha_k^1(|\boldsymbol{z}_k|) \leqslant V_{\circ k}(\boldsymbol{z}_k) \leqslant \alpha_k^2(|\boldsymbol{z}_k|),$$

from which it follows that

$$V_{\circ k}(|\boldsymbol{z}_k - \boldsymbol{z}_i|) \geqslant \alpha_k^1(|\boldsymbol{z}_k - \boldsymbol{z}_i|)$$

hence,

$$V_z(\boldsymbol{z}) \geqslant \sum_{k=1}^{N}\left(\sum_{i=1}^{N}\alpha_k^1(|\boldsymbol{z}_k - \boldsymbol{z}_i|)\right).$$

Now, let $\alpha_m(s) := \min_{1\leqslant k\leqslant N}\{\alpha_k^1(s)\}$; α_m is non-decreasing and, without loss of generality, it may be assumed to be continuous and of class $\mathcal{K}_\infty$ because the functions $\alpha_k^1 \in \mathcal{K}_\infty$ for all k. Therefore,

$$V_z(\boldsymbol{z}) \geqslant \sum_{k=1}^{N}\left(\sum_{i=1}^{N}\alpha_m\big(\,|\boldsymbol{z}_k - \boldsymbol{z}_i|\,\big)\right).$$

Next, let us define $\alpha'_m(s) := \alpha_m\big(\sqrt{s}\big)$, which is also of class $\mathcal{K}_\infty$ hence, after Lemma 1 from the Appendix on p. 37, we obtain

$$V_z(\boldsymbol{z}) \geqslant \sum_{k=1}^{N}\alpha'_m\left(\frac{1}{N}\sum_{i=1}^{N}|\boldsymbol{z}_k - \boldsymbol{z}_i|^2\right).$$

Then, we invoke once more Lemma 1 and, successively, Lemma 2 from the Appendix to obtain

$$\begin{aligned} V_z(\boldsymbol{z}) &\geqslant \alpha'_m\left(\frac{1}{N^2}\sum_{k=1}^{N}\sum_{i=1}^{N}|\boldsymbol{z}_k - \boldsymbol{z}_i|^2)\right) \\ &\geqslant \alpha'_m\left(\frac{2}{N}\,|\boldsymbol{z} - \mathbf{1}_N\otimes\boldsymbol{z}_s|^2\right) = \alpha'_m\left(\frac{2}{N}\,|\boldsymbol{e}_z|^2\right) \end{aligned}$$

so the lower-bound in (2.48) follows with

$$\gamma_1(s) := \alpha'_m\left(\frac{2s^2}{N}\right)$$

which is of class $\mathcal{K}_\infty$. The existence of $\gamma_2 \in \mathcal{K}_\infty$, such that $V_z(\boldsymbol{z}) \leqslant \gamma_2(|\boldsymbol{e}_z|)$ is deduced following similar arguments.

Next, we evaluate the total derivative of $V_z(\boldsymbol{z})$ along trajectories of the system (2.4b), to obtain

$$\dot{V}_z(\boldsymbol{z}) = \sum_{k=1}^{N}\sum_{i=1}^{N}\dot{V}_{\circ k}(\boldsymbol{z}_k - \boldsymbol{z}_i) \tag{2.49}$$

where

$$\begin{aligned} \dot{V}_{\circ k}(\boldsymbol{z}_k - \boldsymbol{z}_i) &= \nabla V_{\circ k}(\boldsymbol{z}_k - \boldsymbol{z}_i)\big[f_k^2(\boldsymbol{z}_k,\boldsymbol{y}_k) - f_i^2(\boldsymbol{z}_i,\boldsymbol{y}_i)\big] \\ &= \nabla V_{\circ k}(\boldsymbol{z}_k - \boldsymbol{z}_i)\left[f_k^2(\boldsymbol{z}_k,\boldsymbol{y}_k) - f_k^2(\boldsymbol{z}_i,\boldsymbol{y}_k) + f_k^2(\boldsymbol{z}_i,\boldsymbol{y}_k) - f_i^2(\boldsymbol{z}_i,\boldsymbol{y}_i)\right] \\ &\leqslant -\bar{\alpha}_k\,|\boldsymbol{z}_k - \boldsymbol{z}_i|^2 + \nabla V_{\circ k}(\boldsymbol{z}_k - \boldsymbol{z}_i)\left[f_k^2(\boldsymbol{z}_i,\boldsymbol{y}_k) - f_i^2(\boldsymbol{z}_i,\boldsymbol{y}_i)\right] + \beta_k \\ &\leqslant -\bar{\alpha}_k\,|\boldsymbol{z}_k - \boldsymbol{z}_i|^2 + \nabla V_{\circ k}(\boldsymbol{z}_k - \boldsymbol{z}_i)\left[f_k^2(\boldsymbol{z}_i,\boldsymbol{y}_k) - f_k^2(\boldsymbol{z}_i,\boldsymbol{y}_i)\right] \\ &\quad + \nabla V_{\circ k}(\boldsymbol{z}_k - \boldsymbol{z}_i)\left[f_k^2(\boldsymbol{z}_i,\boldsymbol{y}_i) - f_i^2(\boldsymbol{z}_i,\boldsymbol{y}_i)\right] + \beta_k \end{aligned} \tag{2.50}$$

Due to the continuity of $\nabla V_{\circ k}$, there exists a constant $C_\circ > 0$ such that for all $|\boldsymbol{x}| \leqslant B_x$ we have, for all $k \in \mathcal{I}$,

$$|\nabla V_{\circ k}(\boldsymbol{z}_k - \boldsymbol{z}_i)| \leqslant C_\circ\,|\boldsymbol{z}_k - \boldsymbol{z}_i|\,.$$

Also, since the functions f_i^2 are locally Lipschitz, there exists a continuous, positive, non-decreasing function $k_z : \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $k \in \mathcal{I}$,

$$\left|f_k^2(\boldsymbol{z}_i,\boldsymbol{y}_k) - f_k^2(\boldsymbol{z}_i,\boldsymbol{y}_i)\right| \leqslant k_z(|\boldsymbol{z}|)\,|\boldsymbol{y}_k - \boldsymbol{y}_i|$$

and for all $\boldsymbol{x}$ such that $|\boldsymbol{x}| \leqslant B_x$,

$$\begin{aligned}\left|f_k^2(\boldsymbol{z}_i, \boldsymbol{y}_k) - f_k^2(\boldsymbol{z}_i, \boldsymbol{y}_i)\right| \;&\leqslant k_z(B_x)\,|\boldsymbol{y}_k - \boldsymbol{y}_i| =: C_z\,|\boldsymbol{y}_k - \boldsymbol{y}_i| \\ &= \; C_z\,|\boldsymbol{y}_k - \boldsymbol{y}_s + \boldsymbol{y}_s - \boldsymbol{y}_i| \\ &\leqslant \; C_z(|\boldsymbol{y}_k - \boldsymbol{y}_s| + |\boldsymbol{y}_s - \boldsymbol{y}_i|) \;=\; 2C_z\,|\boldsymbol{e}_y|\end{aligned}$$

while, from (2.43), we obtain

$$\left|f_k^2(\boldsymbol{z}_i, \boldsymbol{y}_i) - f_i^2(\boldsymbol{z}_i, \boldsymbol{y}_i)\right| \leqslant \Delta_f.$$

Using all the previous bounds in (2.50) we obtain

$$\begin{aligned}\dot{V}_{\circ k}(\boldsymbol{z}_k - \boldsymbol{z}_i) \;&\leqslant\; -\bar{\alpha}_k\,|\boldsymbol{z}_k - \boldsymbol{z}_i|^2 + 2C_\circ\,|\boldsymbol{z}_k - \boldsymbol{z}_i|\left(C_z\,|\boldsymbol{e}_y| + \Delta_f\right) + \beta_k \\ &\leqslant\; -\frac{1}{2}\bar{\alpha}_k\,|\boldsymbol{z}_k - \boldsymbol{z}_i|^2 + \bar{C}\,|\boldsymbol{e}_y|^2 + \Delta_f' + \beta_k, \qquad \forall\, i,\, k \in \mathcal{I},\; \boldsymbol{z}_k \in \mathbb{B}_z^k \quad (2.51)\end{aligned}$$

where $\bar{C} = 4\dfrac{C_\circ^2 C_z^2}{\alpha_k}$ and $\Delta_f' = \dfrac{4C_\circ^2\Delta_f^2}{\alpha_k}$. In turn, it follows that

$$\begin{aligned}\dot{V}_z(\boldsymbol{z}) \;&\leqslant\; -\sum_{k=1}^{N}\sum_{i=1}^{N}\left(\frac{1}{2}\alpha_k\,|\boldsymbol{z}_k - \boldsymbol{z}_i|^2 - \bar{C}\,|\boldsymbol{e}_y|^2 - \Delta_f' + \beta_k\right) \\ &\leqslant\; -N\alpha\,|\boldsymbol{z} - \mathbf{1}_N \otimes \boldsymbol{z}_s|^2 + \bar{C}N^2\,|\boldsymbol{e}_y|^2 + N^2\Delta_f' + \sum_{k=1}^{N} N\beta_k \\ &\leqslant\; -N\alpha\,|\boldsymbol{e}_z|^2 + \bar{C}N^2\,|\boldsymbol{e}_y|^2 + N^2\Delta_f' + \bar{\beta}, \qquad (2.52)\end{aligned}$$

where, for the second step, we invoked Lemma 2 from the Appendix and we defined $\alpha = \min\limits_{1\leqslant k\leqslant N} \bar{\alpha}_k$, $\bar{\beta} = \sum_{k=1}^N N\beta_k$.

Finally, we define the Lyapunov function $V(\boldsymbol{e}) = V_y(\boldsymbol{e}_y) + V_z(\boldsymbol{z})$ which, in view of Claim 1, satisfies

$$\frac{1}{2}\,|\boldsymbol{e}_y|^2 + \gamma_1(|\boldsymbol{e}_z|) \leqslant V(\boldsymbol{e}) \leqslant \frac{1}{2}\,|\boldsymbol{e}_y|^2 + \gamma_2(|\boldsymbol{e}_z|),$$

which is equivalent to $\tilde{\gamma}_1(|\boldsymbol{e}|) \leqslant V(\boldsymbol{e}) \leqslant \tilde{\gamma}_2(|\boldsymbol{e}|)$ for an obvious choice of $\tilde{\gamma}_1,\, \tilde{\gamma}_2 \in \mathcal{K}_\infty$. On the other hand, from (2.46) and (2.52), we obtain

$$\dot{V}(\boldsymbol{e}) \leqslant -\alpha N\,|\boldsymbol{e}_z|^2 - \frac{1}{2}\big[\sigma\lambda_2(L) - 2\bar{C}N^2\big]\,|\boldsymbol{e}_y|^2 + N^2\Delta_f' + \frac{C_2}{2} + \beta_k.$$

Therefore, there exists $\sigma^* > 0$ such that for all $\sigma \geqslant \sigma^*$,

$$\dot{V}(\boldsymbol{e}) \leqslant -\alpha N\,|\boldsymbol{e}|^2 + N^2\Delta_f' + \frac{C_2}{2} + \bar{\beta}.$$

Moreover, the latter also holds along trajectories since they are uniformly ultimately bounded. Invoking Proposition 1 with $\mathcal{Z} = \{0\}$, we conclude that for any $\epsilon > 0$ there exists $T > 0$ and $\sigma > \sigma^*$ such that for all $t \geqslant T$

$$|\boldsymbol{e}(t, \boldsymbol{x}_\circ)| \leqslant r + \epsilon.$$

4.2 On practical stability of the collective network behavior

Now we analyze the behavior of the average unit, whose dynamics is given by the equations (2.41a). In our framework, we naturally assume that the nominal dynamics of average-unit ($i.e.$, with $\boldsymbol{e} = 0$) has a stable compact attractor $\mathcal{A}$ and we establish that the stability properties of this attractor are preserved under the network interconnection, albeit, slightly weakened.

A3 *For the system (2.12), there exists a compact invariant set $\mathcal{A} \subset \mathbb{R}^n$ which is asymptotically stable with a domain of attraction $\mathcal{D} \subset \mathbb{R}^n$. Moreover, we assume that there exists a continuously differentiable Lyapunov function $V_{\mathcal{A}} : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ and functions $\alpha_i \in \mathcal{K}_\infty$, $i \in \{1, \dots, 4\}$ such that for all $\boldsymbol{x}_e \in \mathcal{D}$ we have*

$$\alpha_1(|\boldsymbol{x}_e|_{\mathcal{A}}) \leqslant V_{\mathcal{A}}(\boldsymbol{x}_e) \leqslant \alpha_2(|\boldsymbol{x}_e|_{\mathcal{A}}) \tag{2.53a}$$

$$\dot{V}_{\mathcal{A}}(\boldsymbol{x}_e) \leqslant -\alpha_3(|\boldsymbol{x}_e|_{\mathcal{A}}) \tag{2.53b}$$

$$\left|\frac{\partial V_{\mathcal{A}}}{\partial \boldsymbol{x}_e}\right| \leqslant \alpha_4(|\boldsymbol{x}_e|). \tag{2.53c}$$

The second part of the assumption (the existence of V) is purely technical whereas the first part is essential to analyse the emergent synchronised behaviour as well as the synchronisation properties of the network, recasted as a (robust) stability problem. The following statement applies to the general case of diffusively coupled networks.

Theorem 2 *For the system (2.10) assume that the solutions are globally ultimately bounded and Assumptions* **A1**, **A3** *hold. Then, there exist a non-decreasing function $\gamma : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ and, for any $R, \epsilon > 0$ there exists $T^* = T^*(R, \epsilon)$, such that for all $t \geqslant T^*$ and all $\boldsymbol{x}_\circ$ such that $|\boldsymbol{x}_\circ|_{\mathcal{A}} \leqslant R$,*

$$|\boldsymbol{x}_s(t, \boldsymbol{x}_\circ)|_{\mathcal{A}} \leqslant \gamma(\Delta_f, R) + \epsilon. \tag{2.54}$$

Proof. Global ultimate boundedness of the solutions implies that there exists $T^* > 0$ such that $|\boldsymbol{x}(t, \boldsymbol{x}_\circ)| \leqslant B_x$ for all $t \geqslant T^*$; therefore, a similar bound is valid for $\boldsymbol{x}_s$:

$$|\boldsymbol{x}_s(t, \boldsymbol{x}_\circ)| = \left|\frac{1}{N}\sum_{i=1}^{N} \boldsymbol{x}_i(t, \boldsymbol{x}_\circ)\right| \leqslant \frac{1}{N}\sum_{i=1}^{N} |\boldsymbol{x}_i(t, \boldsymbol{x}_\circ)| \leqslant B'_x.$$

This and Assumption **A1** imply the statement of Theorem 1; in particular, (2.42) holds. Hence, using (2.32) we obtain $|G_s(\boldsymbol{e}(t), \boldsymbol{x}_s(t))| \leqslant B_g$ where $B_g := k\big(\beta(\Delta_f), B'_x\big)\beta(\Delta_f)$.

Next, let Assumption **A3** generate a Lyapunov function $V_{\mathcal{A}}$ for the nominal system (2.16) and functions $\alpha_i \in \mathcal{K}_\infty$, satisfying (2.53), hence,

$$\begin{gathered}\alpha_1(|\boldsymbol{x}_s|_{\mathcal{A}}) \leqslant V_{\mathcal{A}}(\boldsymbol{x}_s) \leqslant \alpha_2(|\boldsymbol{x}_s|_{\mathcal{A}}) \\ \dot{V}_{\mathcal{A}}(\boldsymbol{x}_s) \leqslant -\alpha_3(|\boldsymbol{x}_s|_{\mathcal{A}}) \\ |\nabla V_{\mathcal{A}}(\boldsymbol{x}_s)| \leqslant \alpha_4(|\boldsymbol{x}_s|)\end{gathered}$$

and the total derivative of $V_{\mathcal{A}}$ yields, using (2.31),

$$\begin{aligned}\dot{V}_{\mathcal{A}}(\boldsymbol{x}_s) &= \nabla V_{\mathcal{A}}(\boldsymbol{x}_s)^\top f_s(\boldsymbol{x}_s) + \nabla V_{\mathcal{A}}(\boldsymbol{x}_s)^\top f_s(\boldsymbol{x}_s) G_s(\boldsymbol{e}, \boldsymbol{x}_s) \\ &\leqslant -\alpha_3(|\boldsymbol{x}_s|_{\mathcal{A}}) + \alpha_4(|\boldsymbol{x}_s|)\,|G_s(\boldsymbol{e}, \boldsymbol{x}_s)|\end{aligned} \tag{2.55}$$

which implies that

$$\dot{V}_{\mathcal{A}}(\boldsymbol{x}_s(t)) \leqslant -\alpha_3(|\boldsymbol{x}_s(t)|_{\mathcal{A}}) + \alpha_4(B'_x)B_g. \tag{2.56}$$

Strictly speaking, the bound (2.56) holds for all $t \geqslant T^*$ however, in view of forward completeness, the trajectories are bounded on $[0, T^*]$ therefore, (2.56) holds for all $t \geqslant 0$ by redefining, if necessary, the constants B'_x and B_g in function of R that is, of the ball of initial conditions. Then, invoking Proposition 1, we obtain

$$|\boldsymbol{x}_s(t, \boldsymbol{x}_\circ)|_{\mathcal{A}} \leqslant \alpha_1^{-1} \circ \alpha_2 \circ \alpha_3^{-1} \left(\alpha_4 \Big(B'_x(R) \Big) B_g(\Delta_f, R) \right) + \epsilon.$$

The statement follows with

$$\gamma(s_1, s_2) := \alpha_1^{-1} \circ \alpha_2 \circ \alpha_3^{-1} \left(\alpha_4 \Big(B'_x(s_2) \Big) B_g(s_1, s_2) \right).$$

■

In the case that the network is state practically synchronised, it follows that the set $\mathcal{A}$ is practically stable for the network (2.10).

Corollary 3 *Consider the system (2.10) under Assumption* **A3**. *If the set $\mathcal{S}_x$ is practically uniformly globally asymptotically stable for this system, then the attractor $\mathcal{A}$ defined in Assumption* **A3** *is practically asymptotically stable for the average unit (2.31).*

5 Example

To illustrate our theoretical findings we present a brief case-study of analysis of interconnected heterogeneous systems via diffusive coupling. We consider three chaotic oscillators, two of the well-known Lorenz type –see [55], with different parameters, and a Lü system [59]. The dynamics equations are

LORENZ:

$$\begin{aligned} \dot{x}_i &= \gamma_i(y_i - x_i), \quad i = 1, 2 \\ \dot{y}_i &= r_i x_i - y_i - x_i z_i \\ \dot{z}_i &= x_i y_i - b_i z_i \end{aligned}$$

LÜ:

$$\begin{aligned} \dot{x}_3 &= -\frac{\alpha\beta}{\alpha + \beta} x_3 - 2y_3 z_3 + \delta \\ \dot{y}_3 &= \alpha y_3 + x_3 z_3 \\ \dot{z}_3 &= \beta z_3 + x_3 y_3. \end{aligned}$$

A direct computation, using (2.11), shows that the corresponding emergent dynamics for these systems is given by

EMERGENT DYNAMICS:

$$\begin{aligned} 3\dot{x}_e &= -[\gamma_1 + \gamma_2 + \frac{\alpha\beta}{\alpha + \beta}]x_e + [\gamma_1 + \gamma_2 - 2z_e]y_e + \delta, \\ 3\dot{y}_e &= [r_1 + r_2]x_e - [2 - \alpha]y_e \\ 3\dot{z}_e &= 3x_e y_e + [\beta - b_1 - b_2]z_e. \end{aligned}$$

The values of the parameters of the three systems are fixed in order for them to exhibit a chaotic behaviour:

$\gamma_1 = 10$	$\gamma_2 = 16$	$\alpha = -10$
$r_1 = 45.6$	$r_2 = 99.96$	$\beta = -4$
$b_1 = 4$	$b_2 = 8/3$	$\delta = 10$

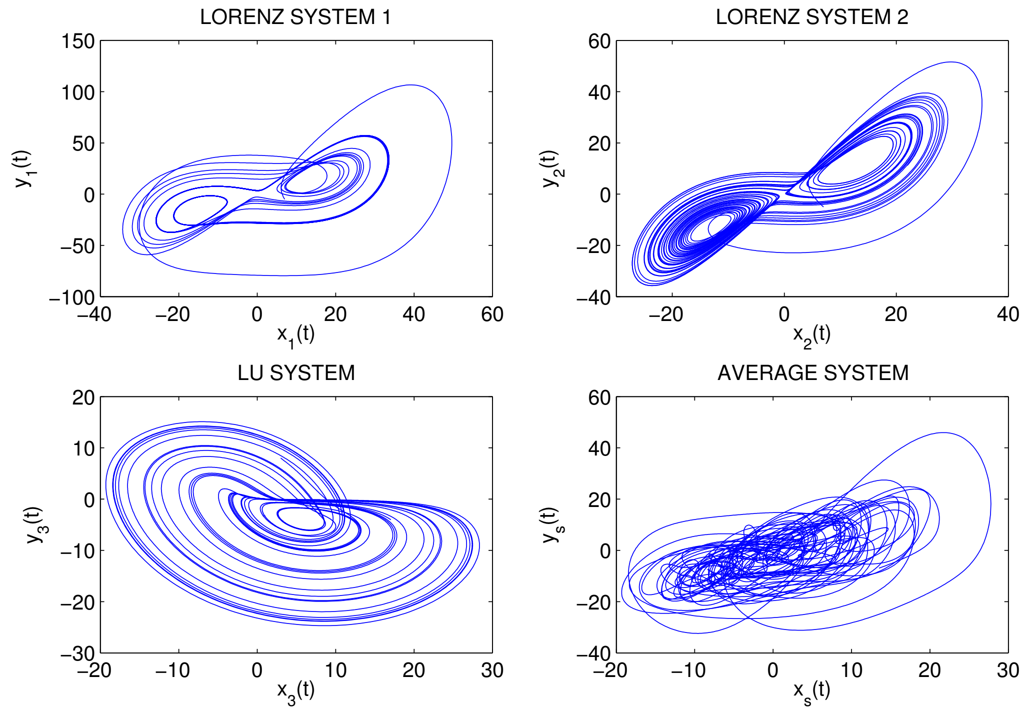


Figure 2.1: Phase portraits of the three chaotic oscillators as well as that of the average dynamics, in the absence of interconnection, *i.e.*, with $\sigma = 0$.

Since the three chaotic systems are *oscillators* their trajectories are globally ultimately bounded –see Figure 2.1. Moreover, as it may be appreciated from Figure 2.2, the solutions remain bounded under the diffusive coupling which, for this test, we defined to be:

$$\begin{aligned} u_1 &= -\sigma\big[d_{12}(x_1 - x_2) + d_{13}(x_1 - x_3)\big], \quad d_{12} = 2,\ d_{13} = 4, \\ u_2 &= -\sigma\big[d_{21}(x_2 - x_1) + d_{23}(x_2 - x_3)\big], \quad d_{23} = 3, \\ u_3 &= -\sigma\big[d_{31}(x_3 - x_1) + d_{32}(x_3 - x_2)\big]. \end{aligned}$$

That is, the zero dynamics with respect to the output $\boldsymbol{y}_i = x_i$ has dimension two. For each Lorenz system, the zero dynamics is practically convergent (Assumption **A1** holds), as it may be showed using the function

$$V(\boldsymbol{z}_i - \boldsymbol{z}_i') = \left|\boldsymbol{z}_i - \boldsymbol{z}_i'\right|^2, \qquad \boldsymbol{z}_i = [y_i\, z_i]^\top, \quad i \in \{1, 2\}.$$

whose total derivative yields

$$\dot{V}(\boldsymbol{z}_i - \boldsymbol{z}_i') \leqslant -2\min\{b_i, 2\}\left|\boldsymbol{z}_i - \boldsymbol{z}_i'\right|^2.$$

For the Lü system, we have, defining $\boldsymbol{z} = [y_3\, z_3]^\top$,

$$\dot{V}(\boldsymbol{z} - \boldsymbol{z}') \leqslant -2\alpha|y_3 - y_3'|^2 - 2\beta|z_3 - z_3'|^2 + 4|x_3(t)||y_3(t) - y_3'(t)||z_3(t) - z_3'(t)|.$$

Convergence in a practical sense (Assumption **A1**) holds since the trajectories are ultimately bounded hence, so is the last term on the right-hand side of the previous inequality.

Simulation results for different values of the interconnection gain σ are showed in Figure 2.2; it may be appreciated that the the synchronisation errors $\boldsymbol{e}_y(t)$ diminish as the interconnection gain is increased. The phase portraits of the three oscillators and that of the average dynamics, are also showed for three different values of σ.

In Figure 2.3 we show the phase portrait of the average dynamics (2.41a), for different values of the interconnection gain, compared to that of the emergent dynamics (2.12). As it is appreciated, the solutions generated by the emergent dynamics converge to an equilibrium –approximately, the point $(2.9, 29.43)$ that is, in this case the attractor $\mathcal{A}$, defined in Assumption **A3**, is a point in the phase space. The average dynamics possesses a double scroll attractor for "small" values of σ and it becomes a point in the space at short distance from $\mathcal{A}$, roughly for $\sigma > 15$. This illustrates that the emergent dynamics constitutes, to some extent, a good "estimate" of the network's collective behaviour.

Finally, the lower-right plot in Figure 2.3 depicts $|\boldsymbol{x}_s(t) - \boldsymbol{x}_e(t)| = |\boldsymbol{x}_s(t)|_{\mathcal{A}}$ which corresponds to the difference between the solutions $\boldsymbol{x}_s(t)$ of the average system (2.41a) and $\boldsymbol{x}_e(t)$, solution of the emergent dynamics $\dot{\boldsymbol{x}}_e = f_s(\boldsymbol{x}_e)$. Note that this difference diminishes as the interconnection gain increases however, it does not vanish as $\sigma \to \infty$ –see (2.54).

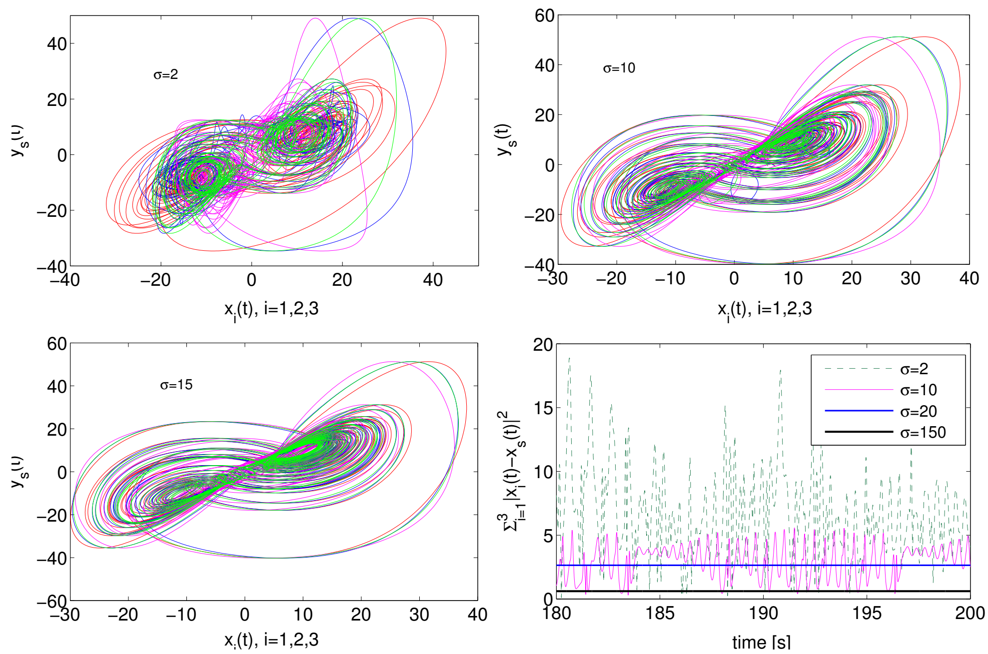


Figure 2.2: First three plots: phase portraits of the three chaotic oscillators compared to that of the average unit, for different values of the interconnection gain σ. In all the plots the ordinates axes refer to $y_s(t)$. The lower-right plot depicts the synchronisation errors $|\boldsymbol{e}_y(t)|$.

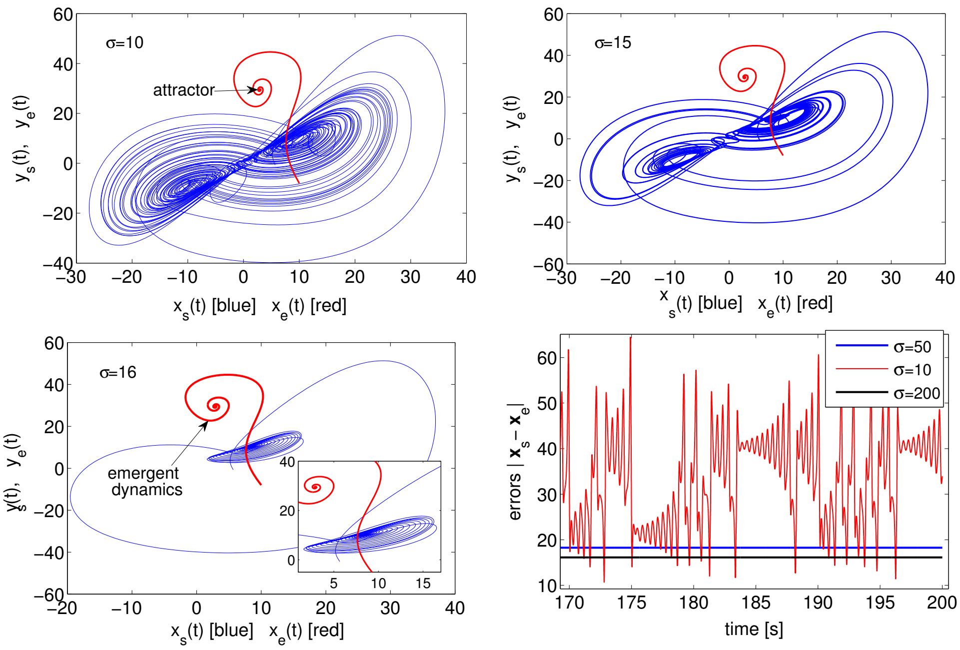


Figure 2.3: x-y phase portraits corresponding to the emergent dynamics (2.12) and the average dynamics (2.41a) for different values of the interconnection gain σ. Bottom-right: $|\boldsymbol{x}_s(t)|_{\mathcal{A}}$.

Appendix: Technical lemmas

Lemma 1 *Let s_i, with $i \in \mathcal{I}$ be arbitrary non-negative scalars, then for any class $\mathcal{K}_\infty$ function $\alpha(\cdot)$, we have*

$$\alpha\left(\frac{1}{N}\sum_{i=1}^{N} s_i\right) \leqslant \sum_{i=1}^{N} \alpha(s_i) \leqslant N\alpha\left(\sum_{i=1}^{N} s_i\right) \tag{2.57}$$

Proof. Let $s_k = \max_{i=1,\ldots,N}\{s_i\}$, then we have that $\frac{1}{N}\sum_{i=1}^{N} s_i \leqslant s_k$ and therefore

$$\alpha\left(\frac{1}{N}\sum_{i=1}^{N} s_i\right) \leqslant \alpha(s_k)$$

so the first inequality holds. Similarly, for the second inequality we have

$$\sum_{i=1}^{N} \alpha(s_i) \leqslant N\alpha(s_k).$$

■

Lemma 2 *Let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N \in \mathbb{R}^n$, $\boldsymbol{x} = [\boldsymbol{x}_1^\top, \boldsymbol{x}_2^\top, \ldots, \boldsymbol{x}_N^\top]^\top \in \mathbb{R}^{nN}$ and $\boldsymbol{x}_s = \dfrac{1}{N}\displaystyle\sum_{i=1}^{N} \boldsymbol{x}_i$. Then,*

$$|\boldsymbol{x} - \mathbf{1}_N \otimes \boldsymbol{x}_s|^2 = \frac{1}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N} |\boldsymbol{x}_i - \boldsymbol{x}_j|^2 .$$

Proof. Let us define the vectors $\boldsymbol{X}, \tilde{\boldsymbol{X}}$ and $\boldsymbol{Y} \in \mathbb{R}^{nN^2}$,

$$\boldsymbol{X} = \mathbf{1}_N \otimes \boldsymbol{x} = \begin{bmatrix} \boldsymbol{x} \\ \vdots \\ \boldsymbol{x} \end{bmatrix}, \qquad \boldsymbol{Y} = \begin{bmatrix} \mathbf{1}_N \otimes \boldsymbol{x}_1 \\ \vdots \\ \mathbf{1}_N \otimes \boldsymbol{x}_N \end{bmatrix}, \quad \tilde{\boldsymbol{X}} = \boldsymbol{X} - \boldsymbol{Y} = \begin{bmatrix} \boldsymbol{x} - \mathbf{1}_N \otimes \boldsymbol{x}_1 \\ \vdots \\ \boldsymbol{x} - \mathbf{1}_N \otimes \boldsymbol{x}_N \end{bmatrix}. \tag{2.58}$$

We establish the statement by showing that the following identities hold:

$$\tilde{\boldsymbol{X}}^\top \tilde{\boldsymbol{X}} = 2N\,|\boldsymbol{x} - \mathbf{1}_N \otimes \boldsymbol{x}_s|^2 ; \tag{2.59}$$

$$\tilde{\boldsymbol{X}}^\top \tilde{\boldsymbol{X}} = \sum_{i=1}^{N}\sum_{j=1}^{N} |\boldsymbol{x}_i - \boldsymbol{x}_j|^2 ; \tag{2.60}$$

Proof of Identity (2.59). By direct calculation we find that

$$\tilde{\boldsymbol{X}}^\top \tilde{\boldsymbol{X}} \;\; = \;\; \boldsymbol{X}^\top \boldsymbol{X} + \boldsymbol{Y}^\top \boldsymbol{Y} - 2\boldsymbol{X}^\top \boldsymbol{Y}.$$

Now, in view of (2.58) the first term on the right hand side satisfies

$$\boldsymbol{X}^\top \boldsymbol{X} = N\,|\boldsymbol{x}|^2 .$$

Then, for the second term, we have

$$\begin{aligned} \boldsymbol{Y}^\top \boldsymbol{Y} &= \sum_{i=1}^{N} (\mathbf{1}_N^\top \otimes \boldsymbol{x}_i)(\mathbf{1}_N \otimes \boldsymbol{x}_i) \\ &= \sum_{i=1}^{N} \mathbf{1}_N^\top \mathbf{1}_N \boldsymbol{x}_i^\top \boldsymbol{x}_i = N\,|\boldsymbol{x}|^2 \,. \end{aligned} \tag{2.61}$$

Finally, for the last term, we have

$$\boldsymbol{X}^\top \boldsymbol{Y} = \sum_{i=1}^{N} \boldsymbol{x}^\top (\mathbf{1}_N \otimes \boldsymbol{x}_i) \tag{2.62}$$

and, for each i, we have

$$\boldsymbol{x}^\top (\mathbf{1}_N \otimes \boldsymbol{x}_i) = \sum_{k=1}^{N} \boldsymbol{x}_k^\top \boldsymbol{x}_i = N \boldsymbol{x}_s^\top \boldsymbol{x}_i$$

therefore,

$$\boldsymbol{X}^\top \boldsymbol{Y} = N \boldsymbol{x}_s^\top \left(\sum_{i=1}^{N} \boldsymbol{x}_i \right) = N^2\,|\boldsymbol{x}_s|^2 \,.$$

Thus, combining the expressions for $\boldsymbol{X}^\top \boldsymbol{X}$, $\boldsymbol{Y}^\top \boldsymbol{Y}$ and $\boldsymbol{X}^\top \boldsymbol{Y}$ we obtain

$$\tilde{\boldsymbol{X}}^\top \tilde{\boldsymbol{X}} = 2N \boldsymbol{x}^\top \boldsymbol{x} - 2N^2 \boldsymbol{x}_s^\top \boldsymbol{x}_s. \tag{2.63}$$

On the other hand, we have

$$\begin{aligned} |\boldsymbol{x} - \mathbf{1}_N \otimes \boldsymbol{x}_s|^2 &= (\boldsymbol{x}^\top - \mathbf{1}_N^\top \otimes \boldsymbol{x}_s^\top)(\boldsymbol{x} - \mathbf{1}_N \otimes \boldsymbol{x}_s) \\ &= \boldsymbol{x}^\top \boldsymbol{x} - 2\boldsymbol{x}^\top (\mathbf{1}_N \otimes \boldsymbol{x}_s) + (\boldsymbol{x}_s^\top \boldsymbol{x}_s)(\mathbf{1}_N^\top \mathbf{1}_N) \\ &= \boldsymbol{x}^\top \boldsymbol{x} - N \boldsymbol{x}_s^\top \boldsymbol{x}_s. \end{aligned} \tag{2.64}$$

so (2.59) follows by comparing (2.63) and (2.64).

Proof of Identity (2.60). By definition of $\tilde{\boldsymbol{X}}$ we have

$$\tilde{\boldsymbol{X}}^\top \tilde{\boldsymbol{X}} = \sum_{i=1}^{N} |\boldsymbol{x} - \mathbf{1}_N \otimes \boldsymbol{x}_i|^2 \,,$$

while, by direct computation, we see that

$$\left[\boldsymbol{x} - \mathbf{1}_N \otimes \boldsymbol{x}_i\right]^\top \left[\boldsymbol{x} - \mathbf{1}_N \otimes \boldsymbol{x}_i\right] = \sum_{k=1}^{N} |\boldsymbol{x}_k - \boldsymbol{x}_i|^2$$

so (2.60) follows. ∎

CHAPTER 3

Synchronisation of Oscillators Networks

1 Introduction

In the previous chapter we have seen that one of the key issues in synchronisation analysis in networks of heterogeneous systems pertains to the emergence of new collective phenomena and the manner how individual dynamics, as well as the coupling architecture, affect the arising synchronised dynamics. For instance, the collective behaviour of a network of coupled nonlinear oscillators is important to understand the complex dynamics of some engineering and physical systems.

From a historical point of view, the first mathematical formulation of the synchronisation problem for nonlinear oscillators is due to A. Andronov [2], although N. Wiener observed this phenomenon in nature and was the first to recognise its importance in the generation of characteristic rhythms in the brain [89]. Following the results of Andronov on limit-cycle oscillators, more generalised versions of the coupled oscillators model, including both phase and amplitude variations, have been published. Among these, the complex Stuart-Landau equation displays the amplitude equation derived from a general ordinary differential equation near an Andronov-Hopf bifurcation point –[82]. The Stuart-Landau oscillator is used in a wide range of applications; for instance, to describe chemical reaction diffusion systems [40], semiconductor lasers [16] as well as in neuro-physiology [6].

From a generic analytical viewpoint, issues related to dependency of the synchronisation on the coupling strength among the oscillators and the stability properties of the synchronous dynamics, have attracted steadily increasing attention during the last few decades. It is well known that a network of such oscillators synchronises for the large values of interconnection gains. Based on this property, and assuming that in the limit (in terms of coupling gain) all oscillators have the same limit cycle, in the 1970s Y. Kuramoto proposed a reduced order model which characterise limit case behaviour of such oscillators which became one of the most popular models of phase oscillators and is now known as the Kuramoto model. This model exhibits cooperative phenomena, such as frequency synchronisation and phase-locking of the oscillators beyond a certain coupling strength [76], [49], [81].

In the case of finite gain and non-identical individual dynamics, $i.e.$, the case of heterogeneous networks, the coupled Stuart-Landau oscillators are only frequency synchronised that is, the amplitudes of their oscillations do not coincide. Different tools, such as Dula's theorem and Lyapunov exponents, have been used in the literature to study stability properties of the limit cycle for a single Stuart-Landau oscillator [2], [68], [50] and for networks of such oscillators, see $e.g.$, [45], [86], Lyapunov type techniques were used to study stability for a network of identical oscillators [69]. However, in the general case of heterogeneous networks of Stuart-Landau oscillations, finding the synchronisation frequency is a challenging and, to the best of our knowledge, open problem. In this chapter we give an approximate expression for this frequency which depends on the parameters of the individual oscillators and on the matrix of the interconnections.

Furthermore, we present original results on synchronisation of Stuart-Landau oscillators with different oscillating frequencies, under diffusive coupling. The analysis that we carry out relies on the framework of emergent dynamics, presented in the previous chapter. Accordingly, it consists in two main parts: the stability analysis of the average dynamics and that of the synchronisation manifold corresponding to the synchronisation errors. We use a model of Stuart-Landau oscillators expressed in complex scalar coordinates hence, in regard of the framework of the previous chapter, we study a case of state-synchronisation. One of the main technical difficulties in the study of Stuart-Landau oscillators is that the most classical Lyapunov stability tools (as for instance exposed in [47]) are inapplicable since the equilibrium of the system form a compact disconnected set.

For clarity of exposition, the rest of the chapter is organised in three parts. In the next section we study the stability of a single oscillator both for the forced and unforced cases. The latter is fundamental in the study of the stability of the average dynamics, in the context of interconnected oscillators over networks. Stability of the synchronisation manifold is studied next and a brief numerical example is presented at the end of the chapter, for the sake of illustration.

2 Stability of the Stuart-Landau oscillator

We recall, for convenience, that the unforced Stuart-Landau equation, which represents a normal form of the Andronov-Hopf bifurcation, is given by

$$\dot{z} = -\nu \left|z\right|^2 z + \mu z \tag{3.1}$$

where $z \in \mathbb{C}$ denotes the state of the oscillator, ν, $\mu \in \mathbb{C}$ are parameters defined as $\nu = \nu_{\mathsf{R}} + \mathrm{i}\nu_{\mathsf{I}}$ and $\mu = \mu_{\mathsf{R}} + \mathrm{i}\mu_{\mathsf{I}}$.

The analysis of Equation (3.1) is well documented in the literature via, $e.g.$, Lyapunov-exponents methods (see $e.g.$, [50] and [68], for a detailed overview), or using the second method of Lyapunov (see $e.g.$, [62] and [69]). Of particular interest in the study Stuart-Landau oscillators is the case when $\nu_{\mathsf{R}} > 0$ since, otherwise, in the case that $\nu_{\mathsf{R}} < 0$, the solutions of the system may explode in finite time and if $\nu_{\mathsf{R}} = 0$ the oscillator becomes a simple first-order linear system. It is also clear that the origin is unstable if $\mu_{\mathsf{R}} > 0$. Its behaviour on the phase plane is illustrated in Figure 1.1 in Chapter 1.

Remark 2 *In the analysis of (the solutions of) (3.1) we use some statements originally formulated for systems whose state space is Euclidean. In this regard, it is convenient to stress that, for a* *dynamical system* $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$*, with* $\boldsymbol{x} \in \mathbb{C}^N$*, one can define stability in the sense of Lyapunov similarly as for systems whose state-space is restricted to* $\mathbb{R}^N$*. Indeed, for a complex vector* $\boldsymbol{x} = \boldsymbol{x}_R + \mathrm{i}\boldsymbol{x}_I \in \mathbb{C}^N$*, we may define the vector* $\tilde{\boldsymbol{x}} \in \mathbb{R}^{2N}$ *as* $\tilde{\boldsymbol{x}} := [\boldsymbol{x}_R^\top \ \boldsymbol{x}_I^\top]^\top$*. Note that, in particular,* $|\tilde{\boldsymbol{x}}|^2 = |\boldsymbol{x}|^2$*. Then, provided that* f *admits the decomposition* $f(\boldsymbol{x}) := f_R(\boldsymbol{x}_R, \boldsymbol{x}_I) + \mathrm{i} f_I(\boldsymbol{x}_R, \boldsymbol{x}_I)$*, we may re-express the dynamics of* $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$ *in a* $2N$*-dimensional Euclidean space, via*

$$\begin{aligned} \dot{\boldsymbol{x}}_R &= f_R(\boldsymbol{x}_R, \boldsymbol{x}_I) \\ \dot{\boldsymbol{x}}_I &= f_I(\boldsymbol{x}_R, \boldsymbol{x}_I) \end{aligned}$$

and stability of the origin $\{\boldsymbol{x} = 0\} \subset \mathbb{C}^N$ *is equivalent to the stability of* $\{\tilde{\boldsymbol{x}} = 0\} \subset \mathbb{R}^{2N}$*. Consequently, we may safely invoke statements originally formulated for systems on Euclidean spaces, to draw conclusions regarding stability of solutions of systems in the complex (hyper)plane.*

Furthermore, note that the assumption that f *admits the previous factorisation is a mild assumption that holds for (at least once) differentiable functions, in particular polynomials, the exponential function* $etc.$

2.1 Stability of the unforced oscillator

As we have explained in Chapter 1, the set

$$\mathcal{W} := \left\{ z \in \mathbb{C} : |z| = \sqrt{\frac{\mu_{\mathsf{R}}}{\nu_{\mathsf{R}}}} \right\} \bigcup \{z = 0\} \tag{3.2}$$

is invariant for the trajectories of the unforced oscillator (3.1). More precisely, the following theorem generalises a statement from [69] concerning the case of real coefficients, $i.e.$, with $\nu_{\mathsf{R}} = 1$ and $\nu_{\mathsf{I}} = 0$.

Theorem 3 *For the unforced Stuart-Landau oscillator, defined by Equation (3.1), the following statements hold true:*

1. *if* $\mu_R \leqslant 0$ *then the origin* $z \equiv 0$ *is globally exponentially stable;*
2. *if* $\mu_R > 0$ *then the limit cycle* $\mathcal{W}_1 = \{z \in \mathbb{C} : |z| = \sqrt{\mu_R/\nu_R}\}$ *is almost globally asymptotically stable and the origin* $\{z = 0\}$ *is antistable*[1]*. Moreover, in this case, the oscillation frequency on* $\mathcal{W}_1$ *is defined by*
$$\omega = \mu_I - \frac{\nu_I}{\nu_R}\mu_R.$$

Proof of Item 1. Global asymptotic stability of the origin $\{z = 0\}$ may be established using the Lyapunov function candidate $V(z) = |z|^2 = \bar{z}z$ where $\bar{z}$ denotes the conjugate of z. Indeed, taking the derivative of V along trajectories of (3.1) we obtain

$$\begin{aligned} \dot{V}(z) &= \big[-\bar{\nu}\,|z|^2\,\bar{z} + \bar{\mu}\bar{z}\big]z + \bar{z}\big[-\nu\,|z|^2\,z + \mu z\big] \\ &= -(\nu + \bar{\nu})|z|^4 + (\mu + \bar{\mu})\,|z|^2 \\ &= -2\nu_{\mathsf{R}}|z|^4 + 2\mu_{\mathsf{R}}\,|z|^2\,. \end{aligned}$$

[1]That is, the poles of the linearised system have all positive real parts.

Since $\mu_R \leqslant 0$, we have $\dot{V}(z) \leqslant -|\mu_R|\,|z|^2$ for all $z \in \mathcal{C}$ and global exponential stability of the origin follows.

Proof of Item 2. Anti-stability of the origin follows trivially by evaluating the total derivative of $V(z) = |z|^2$ along the trajectories of Equation (3.1) linearised around the origin, $i.e.$, $\dot{z} = \mu z$. Indeed, locally, $\dot{V}(z) = \mu_R\,|z|^2$ where $\mu_R > 0$.

Next, to analyse the stability of the limit cycle $\mathcal{W}_1$, we introduce the Lyapunov function candidate

$$V(z) = \frac{1}{4\nu_R}\big[\,|z|^2 - \alpha\big]^2, \tag{3.3}$$

where $\alpha = \mu_R/\nu_R$. Notice that $V(z) = 0$ for all $z \in \mathcal{W}_1$ and it is positive otherwise.

Evaluating the total derivative of V, along the solutions of (3.1), we get

$$\begin{aligned}\dot{V}(z) &= \frac{1}{2\nu_R}\big[\,|z|^2 - \alpha\big]\big[\dot{\bar{z}}z + \bar{z}\dot{z}\big]\\ &= \frac{1}{2\nu_R}\big[\,|z|^2 - \alpha\big]\big[(-\bar{\nu}\,|z|^2\,\bar{z} + \bar{\mu}\bar{z})z + \bar{z}(-\nu\,|z|^2\,z + \mu z)\big]\end{aligned}$$

and, after regrouping the terms in the last bracket, we obtain

$$\begin{aligned}\dot{V}(z) &= \frac{1}{2\nu_R}\big[\,|z|^2 - \alpha\big]\big[-(\nu + \bar{\nu})|z|^4 + (\mu + \bar{\mu})\,|z|^2\,\big]\\ &= \frac{1}{\nu_R}\big[\,|z|^2 - \alpha\big]\big[-\nu_R\,|z|^2 + \mu_R\big]\,|z|^2\\ &= -\big[\,|z|^2 - \alpha\big]^2\,|z|^2\,.\end{aligned}$$

We conclude that $\dot{V}$ is negative definite with respect to $\mathcal{W}_1$ that is, $\dot{V} < 0$ for all $z \notin \mathcal{W}_1$ and $\dot{V} = 0$ for all $z \in \mathcal{W}_1$. Since the origin is an antistable equilibrium point, $\mathcal{W}_1$ is almost globally asymptotically stable.

It also follows that $r \to \sqrt{\mu_R/\nu_R}$ hence, after Equation (1.6b) and $\omega = \dot{\varphi}$, we have $\omega \to \mu_I - (\nu_I\mu_R)/\nu_R$. ■

2.2 Stability of the forced Stuart-Landau oscillator

For the case when $\mu_R > 0$, in the previous section we proved that the Stuart-Landau oscillator without input, given by (3.1), presents a limit cycle which is almost globally asymptotically stable. Now, we analyse the stability and robustness of the solutions of a forced generalised Stuart-Landau oscillator, as defined by the equation

$$\dot{z} = -\nu\,|z|^2\,z + \mu z + u \tag{3.4}$$

where $u \in \mathbb{C}$ is an input to the oscillator. That is, we analyse the input-to-state stability of this system, $i.e.$, stability with respect to external disturbances. Furthermore, the notion of *almost input-to-state stability*, introduced in [3] (see also [5]), applies to the case of an equilibrium point which is stable for all initial states except for a set of measure zero. For Stuart-Landau oscillators, for which there exists a disjoint invariant set, not consituted of disjoint equilibria, we use a recently developed refined tool for input-to-state stability with respect to decomposable invariant sets –see [4]. For the sake of clarity we start by putting in context the essential technical tools that we use.

The mathematical setting

The main advantage of the approach introduced in [4] is that it allows to analyse the robustness properties of the complex invariant sets without the use of tools involving manifolds and dimensionality arguments, while being applicable to the case when the invariant set is compact. For the sake of self-containedness, we briefly recall below the essential definitions and statements from [4] which are required for the robustness analysis of (3.4).

Consider a nonlinear system

$$\dot{x} = f(x, d), \tag{3.5}$$

where the map $f : M \times D \to T_x M$ is assumed to be of class $\mathcal{C}^1$, M is an n dimensional $\mathcal{C}^2$ connected and orientable Riemannian manifold without boundary and D is a closed subset of $\mathbb{R}^m$ containing the origin.

Let $\mathcal{W}$ be a compact invariant set containing all α and ω limit sets of the unforced system

$$\dot{x} = f(x, 0)$$

and which admits a finite decomposition without cycles, $i.e.$,

$$\mathcal{W} = \bigcup_{i=1}^{k} \mathcal{W}_i \tag{3.6}$$

where $\mathcal{W}_i$ denote non-empty disjoint compact sets which form a *filtration ordering* of $\mathcal{W}$. According to [4] cycles and filtration ordering are defined as follows. First, we introduce the "domains of attraction" and "repulsion" of a set Λ, respectively, as

$$\begin{aligned} W^s(\Lambda) &:= \left\{ x_\circ \in M : |x(t, x_\circ, d)|_\Lambda \to 0 \text{ as } t \to +\infty \right\} \\ W^u(\Lambda) &:= \left\{ x_\circ \in M : |x(t, x_\circ, d)|_\Lambda \to 0 \text{ as } t \to -\infty \right\}. \end{aligned}$$

Then, for two subsets, $\Lambda \subset M$ and $\Gamma \subset M$, we define the relation $\Lambda \prec \Gamma$ as

$$\Lambda \prec \Gamma \quad \Leftrightarrow \quad W^s(\Lambda) \cap W^u(\Gamma) \neq \varnothing. \tag{3.7}$$

Based on these notations, we say that the decomposition $\mathcal{W}_1, \ldots, \mathcal{W}_k$ of $\mathcal{W}$ presents an r-cycle if there is an ordered r-tuple such that $\mathcal{W}_1 \prec \cdots \prec \mathcal{W}_r \prec \mathcal{W}_1$; a 1-cycle if for some i we have $[W^u(\Lambda_i) \cap W^s(\Lambda_i)] - \Lambda_i \neq \varnothing$. Finally, a filtration ordering is an ordered sequence of sets Λ_i such that $\Lambda_i \prec \Lambda_j$ for $i \leqslant j$.

For the case of the Stuart-Landau oscillator, we have the following. Firstly, $\mathcal{W} \subset \mathbb{C}$ defined in (3.2) is a compact invariant set which contains the α and ω limit sets of (3.4). This set admits the finite decomposition in compact sets:

$$\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2, \qquad \mathcal{W}_1 := \left\{ z \in \mathbb{C} : \, |z| = \sqrt{\frac{\mu_\mathrm{R}}{\nu_\mathrm{R}}} \right\} \qquad \mathcal{W}_2 := \left\{ z = 0 \right\}.$$

Then, we have following for the system (3.4):

- $W^s(\mathcal{W}_1) = \left\{ z_\circ \in \mathbb{C} : |z(t, z_\circ)|_{\mathcal{W}_1} \to 0 \text{ as } t \to +\infty \right\}$. This corresponds to the set of initial conditions generating trajectories which converge to the circumference $\mathcal{W}_1$. Since, according to Theorem 3, $\mathcal{W}_1$ is almost globally asymptotically stable, $W^s(\mathcal{W}_1) = \mathbb{C} - \{0\}$.

- $W^u(\mathcal{W}_1) = \{z_\circ \in \mathbb{C} : |z(t,z_\circ)|_{\mathcal{W}_1} \to 0 \text{ as } t \to -\infty\}$. This corresponds to the set of initial conditions generating trajectories that are repulsed away from the circle $\mathcal{W}_1$ hence, $W^u(\mathcal{W}_1) = \varnothing$ since $\mathcal{W}_1$ is almost globally attractive.
- $W^s(\mathcal{W}_2) = \{z_\circ \in \mathbb{C} : |z|_{\mathcal{W}_2} = |z(t,z_\circ)| \to 0 \text{ as } t \to +\infty\}$. This corresponds to the domain of attraction of the origin, however, we know from the proof of Theorem 3 that $\{0\}$ is antistable hence, $W^s(\mathcal{W}_2) = \varnothing$.
- $W^u(\mathcal{W}_2) = \{z_\circ \in \mathbb{C} : |z(t,z_\circ)|_{\mathcal{W}_2} \to 0 \text{ as } t \to -\infty\}$. This corresponds to the set of initial states generating trajectories which are repulsed away from the origin, hence, it corresponds to the disk whose boundary corresponds to $\mathcal{W}_1$, taken away the origin, *i.e.*, $W^u(\mathcal{W}_2) = \{z \in \mathbb{C} : 0 < |z| < \sqrt{\mu_{\mathsf{R}}/\nu_{\mathsf{R}}}\}$.

We conclude that $\mathcal{W}$ admits the filtration ordering $\mathcal{W}_1 \prec \mathcal{W}_2$ because $[\mathbb{C}-\{0\}] \cap W^u(\mathcal{W}_2) \neq \varnothing$ but it contains no 2-cycle because $\mathcal{W}_2 \nprec \mathcal{W}_1$ since $W^s(\mathcal{W}_2) \cap W^u(\mathcal{W}_1) = \varnothing$. It contains no 1-cycle either because $[W^u(\mathcal{W}_1) \cap W^s(\mathcal{W}_1)] - \mathcal{W}_1 = \varnothing$ and $[W^u(\mathcal{W}_2) \cap W^s(\mathcal{W}_2)] - \mathcal{W}_2 = \varnothing$.

The previous characterisation of decomposable compact invariant sets constitutes a formal framework to establish conditions under which a perturbed system admits an input-to-stable stability Lyapunov function, as defined next.

Definition 2.1 *[4]. We say that a $\mathcal{C}^1$ function $V : M \to \mathbb{R}$ is an input-to-state-stability Lyapunov function for (3.5) if there exist $\mathcal{K}_\infty$ functions α_1, α_2, α and γ, and a non-negative real c such that*

$$\alpha_1(|x|_{\mathcal{W}}) \leqslant V(x) \leqslant \alpha_2(|x|_{\mathcal{W}}) + c, \tag{3.8}$$

the function V is constant on each $\mathcal{W}_i$ and the following dissipation condition holds:

$$DV(x)f(x,d) \leqslant -\alpha(|x|_{\mathcal{W}}) + \gamma(|d|). \tag{3.9}$$

The following statement, which corresponds to a paraphrasis of [4, Theorem 1], serves to establish robust stability of Stuart-Landau oscillators (3.4). Indeed, as we show farther below, Stuart-Landau oscillators admit input-to-state stability Lyapunov functions.

Theorem 4 *Consider the nonlinear system (3.5) and let $\mathcal{W}$ correspond to the union of disjoint compact invariant sets containing all α and ω and limit sets of the unforced system $\dot{x} = f(x,0)$, such that $\mathcal{W}$ admits a filtration ordering without cycles. Then, the following are equivalent:*

- *the system (3.5) possesses the asymptotic gain property, i.e., there exists $\eta \in \mathcal{K}_\infty$ such that, for all $x \in M$ and all measurable essentially bounded inputs d, the solutions of (3.5), with initial conditions $x_\circ$, are defined for all $t \geqslant 0$ and*

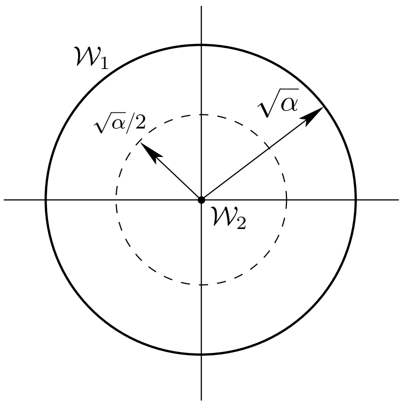
$$\limsup_{t\to+\infty} |x(t,x_\circ,d)|_{\mathcal{W}} \leqslant \eta(\|d\|_\infty) \tag{3.10}$$

  where $\|d\|_\infty := \sup_{t\geqslant 0} |d(t)|$.

- *The system (3.5) admits an input-to-state stability Lyapunov function therefore, it is input to state stable with respect to the input u and the set $\mathcal{W}$.*

Robustness analysis of Stuart-Landau oscillator

We are ready to apply the framework briefly recalled above to analysis of the system (3.4) which, as we have showed, possesses an invariant set decomposable in invariant compacts which admit a filtration ordering with no cycles. These compacts correspond to the (antistable) origin of the complex plane and the almost globally asymptotically stable circle of radius $\sqrt{\mu_R/\nu_R}$. According to Theorem 4, in order to establish input to state stability with respect to $\mathcal{W}$ it is sufficient and necessary to establish that the Stuart-Landau oscillator possesses the asymptotic gain property. To that end, we start by defining the norm $|\cdot|_{\mathcal{W}}$, as follows.



$$|z|_{\mathcal{W}} = \begin{cases} \sqrt{3}\,|z| & \text{if } |z| \leqslant \sqrt{\alpha}/2, \\ \sqrt{\big|\,|z|^2 - \alpha\,\big|} & \text{if } |z| \geqslant \sqrt{\alpha}/2 \end{cases} \qquad (3.11)$$
$$\alpha := \mu_R/\nu_R$$

Figure 3.1: Illustration of $|z|_{\mathcal{W}}$

The following result ensures that the system (3.4) possesses the asymptotic gain property, *i.e.*, asymptotically, the distance between the oscillator's trajectory and set $\mathcal{W}$ becomes proportional to the size of perturbations, $\|d\|_\infty$.

Theorem 5 *Consider the system (3.4) with initial conditions $z_\circ \in \mathbb{C}$ and let the set $\mathcal{W}$ be defined by (3.2). Then, the system (3.4) has the asymptotic gain property, i.e.,*

$$\limsup_{t\to+\infty} |z(t, z_\circ, u)|_{\mathcal{W}} \leqslant \eta\big(\|u\|_\infty\big). \qquad (3.12)$$

Proof. Consider the input-to-state-stability Lyapunov function candidate V defined in (3.3), which we used previously to prove almost global asymptotic stability for the system (3.4). We show next that this function satisfies the inequalities (3.8), (3.9).

Lower bound in (3.8). Let $|z| \leqslant \sqrt{\alpha}/2$ then, $V(z) = |z|^4 - 2\alpha\,|z|^2 + \alpha^2$ and, since $-\,|z|^2 \geqslant -\alpha/4$, we obtain $V(z) = |z|^4 + \alpha/2 \geqslant |z|^4$. Now, since $|z|_{\mathcal{W}} = \sqrt{3}\,|z|$, we obtain $|z| = |z|_{\mathcal{W}}/\sqrt{3}$, which implies that $V(z) \geqslant (1/9)|z|_{\mathcal{W}}^4$.

Next, let $|z| \geqslant \sqrt{\alpha}/2$ then $|z|_{\mathcal{W}} = \sqrt{\Big|\,|z|^2 - \alpha\,\Big|}$

Derivative of V. Evaluating the total derivative of $V(z)$ along trajectories of (3.4) we obtain

$$\dot{V}(z) \;=\; \frac{1}{2\nu_R}\big[\,|z|^2 - \alpha\big]\big[\,(-\bar{\nu}\,|z|^2\,\bar{z} + \bar{\mu}\bar{z})z + \bar{z}(-\nu\,|z|^2\,z + \mu z) + (\bar{u}z + \bar{z}u)\,\big].$$

Now, after the Proof of Item 2 of Theorem 3 and using the triangle inequality, we obtain

$$\begin{aligned}\dot{V}(z) &= -\big[\,|z|^2-\alpha\big]^2|z|^2+\frac{1}{2\nu_{\mathsf{R}}}\big[\,|z|^2-\alpha\big]\big[\bar{u}z+\bar{z}u\big]\\ &\leqslant -\big[\,|z|^2-\alpha\big]^2|z|^2+\frac{1}{\nu_{\mathsf{R}}}\big[\,|z|^2-\alpha\big]|z|\,|u|\\ &\leqslant -\frac{1}{2}\big[\,|z|^2-\alpha\big]^2|z|^2+\frac{1}{2\nu_{\mathsf{R}}^2}|u|^2. \end{aligned}\tag{3.13}$$

Next, we proceed to bound the right-hand side of (3.13) in terms of $|\,z\,|_{\mathcal{W}}$. To bound the term $\big[\,|z|^2-\alpha\big]^2|z|^2$ let us use (3.11) and consider two cases separately.

Case 1. Let $|z|\leqslant\sqrt{\alpha}/2$, *i.e.*, $|z|^2\leqslant\alpha/4$. Then, we have

$$\big[\alpha-|z|^2\,\big]^2\geqslant\big[\alpha-\tfrac{1}{4}\alpha\big]^2$$

hence,

$$-\big[\,|z|^2-\alpha\big]^2|z|^2\leqslant-\frac{9\alpha^2}{16}\,|z|^2$$

and, since $|\,z\,|_{\mathcal{W}}^2=3\,|z|^2$,

$$-\big[\,|z|^2-\alpha\big]^2|z|^2\leqslant-\frac{3}{16}\alpha^2|\,z\,|_{\mathcal{W}}^2$$

Case 2. Let $|z|\geqslant\sqrt{\alpha}/4$ or, equivalently, $|z|^2\geqslant\alpha/4$. Then, we have $|\,z\,|_{\mathcal{W}}=\sqrt{\big|\,|z|^2-\alpha\,\big|}$ and

$$-\big[\,|z|^2-\alpha\big]^2|z|^2\leqslant-\frac{1}{4}\alpha\big[\,|z|^2-\alpha\big]^2=-\frac{1}{4}\alpha|\,z\,|_{\mathcal{W}}^2$$

Combining the two cases together and denoting $c_3=\min\big\{\frac{3}{16}\alpha^2,\ \frac{1}{4}\alpha\big\}$ we obtain

$$\dot{V}(z)\leqslant-c_3|\,z\,|_{\mathcal{W}}^2+\frac{1}{2\nu_{\mathsf{R}}^2}|u|^2,$$

which implies that V is an input-to-state-stability Lyapunov function and, by Theorem 4, we conclude that the system possesses the asymptotic-gain property. ■

3 Synchronisation of networked Stuart-Landau oscillators

In this section we consider a network composed of N heterogeneous diffusively coupled Stuart-Landau oscillators that is, N dynamical systems

$$\begin{aligned}\dot{z}_i &= f(z_i,\mu_i)+u_i,\quad i\in\mathcal{I}:=\{1,\ldots,N\}\\ f(z_i,\mu_i) &:= -|z_i|^2z_i+\mu_iz_i\end{aligned}\tag{3.14}$$

where z_i, $u_i\in\mathbb{C}$ are, respectively, the state and the input of ith oscillator, $\mu_i=\mu_{\mathsf{R}i}+\mathrm{i}\mu_{\mathsf{I}i}\in\mathbb{C}$ is a complex parameter which defines the asymptotic behaviour of the ith oscillator. Heterogeneity of the network is due to the fact that the parameters $\mu_i\in\mathbb{C}$ are different for each oscillator.

We assume that the oscillators are interconnected via diffusive coupling, which represents a static interaction between inputs and states of the oscillators, $i.e.$, for the i-th oscillator the input is given by

$$u_i = -\gamma\Big[d_{i1}(z_i - z_1) + d_{i2}(z_i - z_2)\ldots + d_{iN}(z_i - z_N)\Big], \quad d_{ij} \geqslant 0, \tag{3.15}$$

where the scalar parameter $\gamma > 0$ corresponds to the coupling strength.

In the particular case when oscillators are completely decoupled ($i.e.$, $\gamma = 0$), all the oscillators in the network rotate at their individual (natural) frequencies. Actually, it was shown in [30] that this individual behaviour persists in the case of weak coupling ($i.e.$, for small values of γ). The effect of network synchronisation, which appears in the case of strong coupling, is well documented in the literature; it may be of two types:

- *Frequency synchronisation*: for sufficiently large values of γ all the units tend asymptotically to oscillate at the same frequency, see $e.g.$, [61].
- *Phase locking*: in addition to frequency synchronisation the phase differences between the oscillators tend to be constant and are independent of initial conditions.

In the case of a homogeneous and symmetric network, $i.e.$, in which case $\mu_i = \mu_j$ for all i, $j \in \mathcal{I}$ and $L = L^\top$, all of the systems tend to oscillate at the same frequency and with zero phase differences. As we explained in the previous chapter, this phenomenon, which we call *dynamic consensus*, may be appropriately studied in terms of the asymptotically identical evolution of the units' motions relative to that of the *balanced* average unit's, defined via

$$z_m(t) = \frac{1}{N}\sum_{i=1}^{N} z_i(t). \tag{3.16}$$

In other words, synchronization may be formulated as a problem of asymptotic stability of the synchronization manifold

$$\mathcal{S}_z = \{z_i \in \mathbb{C} : z_1 - z_m = z_2 - z_m \ldots = z_N - z_m = 0\} \tag{3.17}$$

relying on the emergent-dynamics framework described previously. This way, we recover the essence of the results, $e.g.$, in [71, 70, 69].

For heterogeneous networks, we extend the results of the previous chapter. As in the latter, the network's behaviour is completely characterized via the stability of two sets: the attractor of the emergent dynamics and the synchronisation manifold. Nonetheless, the average motion is no longer defined as a balanced average, $i.e.$, by (3.16) and the mean-field variable z_m is scaled in the definition of the synchronisation errors manifold.

The analysis of the former is carried out by studying input to state stability of the average system dynamics with respect to a decomposable compact invariant set while the analysis of the synchronisation manifold is studied by analysing the (practical asymptotic) stability, in the sense of Definition 2.2 on p. 21, of the synchronization errors manifold.

4 Network structure

We assume that the graph of the network is connected and undirected, in which case the interconnections between the nodes are defined by the adjacency matrix $\mathcal{D} := [d_{ij}]_{i,j\in\mathcal{I}_N}$ where $d_{ij} = d_{ji}$ for all $i,\, j \in \mathcal{I}_N$. For simplicity we assume that the interconnections weights are real, $i.e.$, $d_{ij} \in R$ for all $i, j \in \mathcal{I}_N$. Then, the corresponding Laplacian matrix is defined as

$$L = \begin{bmatrix} \sum_{i=2}^{N} d_{1i} & -d_{12} & \dots & -d_{1N} \\ -d_{21} & \sum_{i=1,i\neq 2}^{N} d_{2i} & \dots & -d_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -d_{N1} & -d_{N2} & \dots & \sum_{i=1}^{N-1} d_{Ni}, \end{bmatrix} \tag{3.18}$$

where all row sums are equal to zero. Since the the network is connected and undirected L has exactly one eigenvalue (say, λ_1) equal to zero, while others are positive, $i.e.$, $0 = \lambda_1 < \lambda_2 \leqslant \ldots \leqslant \lambda_N$. Therefore, denoting by $\boldsymbol{z} \in \mathbb{C}^N$ the overall network's state, that is $\boldsymbol{z} = [z_1, \ldots, z_N]^\top$, using (3.14) and the expression for the diffusive coupling, (3.15), we see that the overall network dynamics can be described by the N differential equations

$$\dot{\boldsymbol{z}} = F(\boldsymbol{z}) - \gamma L\boldsymbol{z}, \tag{3.19}$$

where the function $F : \mathbb{C}^N \to \mathbb{C}^N$ is given by

$$F(\boldsymbol{z}) = [f(z_i, \mu_i)]_{i\in\mathcal{I}}\,. \tag{3.20}$$

In order to analyse the behaviour of the solutions, $\boldsymbol{z}(t)$, of (3.19) we proceed to rewrite the system dynamics in new coordinates which exhibit the network emergent dynamics. As we shall prove, the synchronisation properties may be deduced via an eigenvalue analysis of the linear part on the right-hand side of (3.19). To that end, we proceed to underline several structural properties of the networked system (3.19).

To start with, let

$$C(\boldsymbol{z}) := \begin{bmatrix} |\boldsymbol{z}_1|^2 & 0 & \dots & 0 \\ 0 & |\boldsymbol{z}_2|^2 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & |\boldsymbol{z}_N|^2 \end{bmatrix} \quad \text{and} \quad \mathcal{M} := \begin{bmatrix} \mu_1 & 0 & \dots & 0 \\ 0 & \mu_2 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \mu_N \end{bmatrix}$$

then, we may rewrite the system (3.19) as

$$\dot{\boldsymbol{z}} \;=\; A_\gamma \boldsymbol{z} - C(\boldsymbol{z})\boldsymbol{z}, \tag{3.21a}$$

$$A_\gamma \;:=\; \mathcal{M} - \gamma L. \tag{3.21b}$$

The interest of representing the network dynamics as in (3.21) is that it enables us to study the behaviour of the networked oscillators, following relatively simple arguments which rely on matrix and graph theory. A fundamental, non-obvious, fact that we shall exhibit is that A_γ possesses properties similar to those of the Laplacian L. In particular, the eigenvalues of A_γ approach those of L (in absolute value) for large values of the interconnection gain γ. To see this, we express the matrix A_γ as a "perturbed version" of the Laplacian, *i.e.*,

$$A_\gamma = \gamma\left(-L + \varepsilon\mathcal{M}\right), \qquad \varepsilon := \frac{1}{\gamma}$$

in which the parameter $\varepsilon = 1/\gamma$ may be rendered arbitrarily small by design. That is, for sufficiently large values of γ, we may use results on perturbation theory for matrices (see, *e.g.*, [38, 63]) to characterise the eigenvalues and eigenvectors of A_γ in terms of ε and the eigenvalues and eigenvectors of the Laplacian L. In particular, [63, Theorem 2.1] as well as [38], [90] allow to estimate the eigenvalues of A_γ in terms of those of L, $\mathcal{M}$ and ε. In general, a small perturbation of a generic matrix A is denoted by

$$A_\varepsilon = A_0 + \varepsilon A_1, \qquad \varepsilon \to 0 \tag{3.22}$$

so, if we denote by $\lambda_1(A_0)$ a simple eigenvalue of A_0 and by $\lambda_{1\varepsilon}$ its induced perturbation, then, for sufficiently small ε, we may use the convergent power series representation

$$\lambda_\varepsilon = \lambda_1 + c_1\varepsilon + o(\varepsilon), \tag{3.23}$$

where the coefficient of the first-order term, $c_1\varepsilon$, may be characterised as

$$c_1 = \frac{w^\top A_1 v}{w^\top v} \tag{3.24}$$

where w and v are normalised left and right eigenvectors of the unperturbed matrix A_0 associated to λ_1 hence, $|w| = |v| = 1$. This result is also applicable if the multiplicity of λ_1 is larger than one, provided that there exists a complete set of eigenvectors for the associated eigenspace [63], [90].

On the other hand, we underline that $A_\gamma \in \mathbb{C}^{N\times N}$ is complex symmetric, *i.e.*, $A_\gamma = A_\gamma^\top$ and, as it is well known (see *e.g.*, [25, 38]) for any symmetric complex matrix M there exists a complex orthogonal matrix T, *i.e.*, satisfying $T^{-1} = T^\top$, such that $T^\top M T$ has the block-diagonal form

$$\begin{bmatrix} M_1 & 0 & 0 & \dots \\ 0 & M_2 & 0 & \dots \\ 0 & 0 & M_3 & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}$$

where each block M_k is either scalar, if the eigenvalue is simple, or $M_k = \lambda_k I + \tilde{M}$ where $\tilde{M} \in \mathbb{C}^{q\times q}$, if the eigenvalue has multiplicity q, and the eigenvalues of $\tilde{M}_k$ equal to zero.

Now, for the system (3.21a) the Laplacian matrix L is symmetric and corresponds to a connected graph hence, it is diagonalizable and there exists a real orthogonal matrix U such that

$$L = U \begin{bmatrix} \lambda_1(L) & & \\ & \ddots & \\ & & \lambda_N(L) \end{bmatrix} U^\top$$

where, we recall that $\lambda_1(L) = 0$. Moreover, the left and right eigenvectors of L that correspond to $\lambda_1(L) = 0$ coincide and correspond to

$$w = v = \frac{1}{N}\mathbf{1}, \qquad \mathbf{1} := [1 \cdots 1]^\top.$$

Thus, by assimilating A_ε in (3.22) to $(-L+\varepsilon\mathcal{M})$ hence, A_0 to $-L$ and A_1 to $\mathcal{M}$, we see from (3.24), that

$$c_1 = \frac{1}{N}\mathbf{1}^\top\mathcal{M}\mathbf{1} = \frac{1}{N}\sum_{i=1}^{N}\mu_i$$

and we deduce that the eigenvalues of A_γ may be approximated, via (3.23), as

$$\begin{aligned}\lambda_1(A_\gamma) &= \gamma\big[-\lambda_1(L)+c_1\varepsilon+o(\varepsilon)\big]\\ &= \gamma\Big[c_1\frac{1}{\gamma}+o\big(\frac{1}{\gamma}\big)\Big]\\ &= \frac{1}{N}\sum_{i=1}^{N}\mu_i+O\big(\frac{1}{\gamma}\big).\end{aligned}$$

We conclude that $\lambda_1(A_\gamma)$ is bounded as a function of γ and it converges to $\frac{1}{N}\sum_{i=1}^{N}\mu_i$ as the coupling strength $\gamma\to\infty$. Moreover, for all $j\in\{2,\dots,N\}$ we have

$$\lambda_j(A_\gamma) = -\gamma\lambda_j(L)+c_1+O(\varepsilon),$$

where c_1 was defined in (3.24), hence, the eigenvalues of A_j are proportional to γ and, since $\Re e[\lambda_j(L)]>0$, we have $\Re e\big[\lambda_j(A_\gamma)\big]\to-\infty$ as $\gamma\to\infty$. In view of the above, it should be clear that the following (standing) hypothesis is little restrictive. The first part follows by construction and in view of the structure of the Laplacian. The second part, that the largest eigenvalue of A_γ is simple, reasonably follows under the observation that for large values of the interconnection gains, the eigenvalues of A_γ approach (in absolute value) those of L. In view of the above, the following hypothesis comes naturally.

A4 *There exists a number $\gamma^*>0$ and, for each $\gamma\geqslant\gamma^*$, a diagonal matrix $\Lambda_\gamma\in\mathbb{C}^{N\times N}$, whose elements corresponds to the eigenvalues of A_γ, and a complex orthogonal matrix $V_\gamma\in\mathbb{C}^{N\times N}$, $i.e.$, such that*

$$V_\gamma^\top V_\gamma = I_N, \tag{3.25}$$

and the matrix A_γ defined in (3.21b) may be factorised as

$$A_\gamma = V_\gamma\Lambda_\gamma V_\gamma^{-1}. \tag{3.26}$$

Moreover, there exists $k\leqslant N$ such that $\Re e[\lambda_k] > \max_{j\in\mathcal{I}\,\cap\, j\neq k}\Re e[\lambda_j]$.

Without loss of generality, in what follows we assume that the eigenvalues of A_γ are ordered in decreasing order, that is, $\lambda_1(A_\gamma)$ has the largest real part and $\Re e[\lambda_1]>\Re e[\lambda_2]\geqslant\ldots\geqslant\Re e[\lambda_N]$.

We remark that the ith column of the matrix V_γ corresponds to the right eigenvector, denoted $\vartheta_{\mathbf{r}_i}$, associated to the ith eigenvalue of A_γ. Correspondingly, we denote by $\vartheta_{\boldsymbol{\ell}_i}$ the ith left eigenvector, which corresponds to the ith row of $V_\gamma^\top$. Therefore, we have

$$\begin{aligned}A_\gamma\vartheta_{\mathbf{r}_i} &= \lambda_i(A_\gamma)\vartheta_{\mathbf{r}_i},\\ \vartheta_{\boldsymbol{\ell}_i}A_\gamma &= \lambda_i(A_\gamma)\vartheta_{\boldsymbol{\ell}_i}\end{aligned}$$

Moreover, due to the orthogonality of V_γ, expressed by Equation (3.25), we have

$$[\vartheta_{\mathbf{r}}]^2 := \vartheta_{\mathbf{r}}^\top \vartheta_{\mathbf{r}} = 1. \tag{3.27}$$

Another crucial feature of (3.21) is that it leads to a new representation of the dynamics, which is reminiscent of that of a *homogeneous* network. To see this, we proceed to decompose the matrix A_γ as follows. According to Assumption **A4** the matrix matrix Λ_γ is diagonal hence, we may introduce Λ_1, Λ_2 such that

$$\Lambda = \Lambda_1 + \Lambda_2, \quad \Lambda_1 := \lambda_1(A_\gamma)I \tag{3.28a}$$

$$\Lambda_2 := \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \lambda_2(A_\gamma) - \lambda_1(A_\gamma) & 0 & \vdots \\ 0 & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_N(A_\gamma) - \lambda_1(A_\gamma) \end{pmatrix} \tag{3.28b}$$

Notice that if $\gamma > \gamma^*$, where γ^* satisfies Assumption **A4** then $(N-1)$ non-zero eigenvalues of the matrix Λ_2 have negative real parts and, moreover, for all $i \in \{2, \ldots, N\}$ we have $\Re e[\lambda_i(\Lambda_2)] \to -\infty$ as $\gamma \to +\infty$.

Using these notations we can rewrite the matrix A_γ as

$$A_\gamma = V_\gamma \Lambda_1 V_\gamma^\top + V_\gamma \Lambda_2 V_\gamma^\top = \lambda_1(A_\gamma)I + D, \tag{3.29}$$

where $D = V_\gamma \Lambda_2 V_\gamma^\top$. The interest of the matrix D is that it depends on the systems' parameters μ_i but it inherits the properties of the Laplacian matrix; indeed, in view of the definition of Λ_2 and Assumption **A4** we have $D \leqslant 0$ and, moreover, $N-1$ eigenvalues of this matrix have negative real parts. As a matter of fact, for all $i \in \{2, \ldots, N\}$, we have

$$\lambda_i(D) = \lambda_i(\Lambda_2), \ \Re e[\lambda_i(\Lambda_2)] \to -\infty \quad \text{as} \quad \gamma \to +\infty.$$

Moreover, the right eigenvectors $\vartheta_{\mathbf{r}_i}$ associated to the eigenvalues $\lambda_i(A_\gamma)$ of A_γ are also the respective right eigenvectors associated to the eigenvalues of D, $\lambda_i(D) = \lambda_i(A_\gamma) - \lambda_1(A_\gamma)$. Indeed, we have, for each $i \in \mathcal{I}$, $D\vartheta_{\mathbf{r}_i} = V_\gamma \Lambda_2 V_\gamma^\top \vartheta_{\mathbf{r}_i}$. On the other hand, since $\vartheta_{\mathbf{r}_i}^\top \vartheta_{\mathbf{r}_j} = 0$ for all $i \neq j$ and $\vartheta_{\mathbf{r}_i}^\top \vartheta_{\mathbf{r}_i} = 1$, we have

$$\Lambda_2 V_\gamma^\top \vartheta_{\mathbf{r}_i} = \begin{bmatrix} 0 \\ \vdots \\ \lambda_i(A_\gamma) - \lambda_1(A_\gamma) \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \end{matrix}\right\} i-1 \text{ zeroes} \\ \\ \\ \\ \end{matrix}$$

therefore, $V_\gamma \Lambda_2 V_\gamma^\top \vartheta_{\mathbf{r}_i} = \vartheta_{\mathbf{r}_i}[\lambda_i(A_\gamma) - \lambda_1(A_\gamma)]$ that is, $D\vartheta_{\mathbf{r}_i} = \vartheta_{\mathbf{r}_i}\lambda_i(D)$. Clearly, since $\lambda_1(D) = 0$ we also have $D\vartheta_{\mathbf{r}_1} = 0$.

The overall conclusion is that the networked system (3.21a) may be expressed in the alternative form

$$\dot{\boldsymbol{z}} = \big[\lambda_1 I - C(\boldsymbol{z})\big]\boldsymbol{z} + D\boldsymbol{z} \tag{3.30}$$

which is no more than an alternative manner of writing the equations of motion of the interconnected heterogeneous oscillators, (3.19). The interest of this representation is that it is reminiscent of a network in which the oscillators have equal parameters μ_i. Indeed, notice that the dynamics equation for a network (3.14), (3.15) with $\mu_i = \mu_j = \mu$ for all $i,\, j \in \mathcal{I}$ takes the form

$$\dot{\boldsymbol{z}} = \big[\mu I - C(\boldsymbol{z})\big]\boldsymbol{z} - \gamma L \boldsymbol{z}$$

Thus, the fact that D inherits the properties of the Laplacian matrix enables us, to some extent, to interpret the original network of heterogeneous oscillators as a network where all the nodes have identical dynamics.

5 Network dynamics

5.1 Coordinate transformation

Even though the diagonalizability of A_γ allows us to reinterpret the network's equation of motion as that of a homogeneous network, the significance of this property is well beyond pure analytical interest. As we show next, it also allows to *naturally* exhibit the intrinsic emergent dynamics, which is at the core of the networked systems behaviour and, therefore, at the basis of our analysis framework.

To see this clearer, we proceed to represent the system (3.21) in a coordinates frame whose first coordinate corresponds to a certain "average" of all the units' states. The rest of the coordinates, which stem naturally from this representation, correspond to the synchronisation errors. Such coordinate transformation, which is defined upon the transformation matrix V_γ simplifies considerably the analysis of the networked system. Let

$$\tilde{\boldsymbol{z}} = {V_\gamma}^\top \boldsymbol{z} \tag{3.31}$$

and let $\widetilde{V}_\gamma := \big[\vartheta_{\mathbf{r}_2} \cdots \vartheta_{\mathbf{r}_N}\big]$ then,

$$\tilde{\boldsymbol{z}} = \begin{bmatrix} \vartheta_{\mathbf{r}_1}^\top \\ \widetilde{V}_\gamma^\top \end{bmatrix} \boldsymbol{z}. \tag{3.32}$$

From Section 4 we know that $\lambda_1(A_\gamma) \to \lambda_1(L)$ as $\gamma \to \infty$ and, $\vartheta_{\mathbf{r}_1}$, which corresponds to the first right eigenvector of both, A_γ and D, satisfies $\vartheta_{\mathbf{r}_1} \to \mathbf{1}$, as $\gamma \to \infty$. It follows that, in the limit, the coordinate $\tilde{z}_1 := \vartheta_{\mathbf{r}_1}^\top \boldsymbol{z}$ converges to the vector

$$z_e = \frac{1}{N}\sum_{i=1}^{N} z_i$$

which in the literature on nonlinear oscillators is referred to as the state of the *averaged* or *mean-field* oscillator –see [75, 9]. In other words, $\tilde{z}_1$ may be regarded as a weighted average of the units' states z_i.

Next, let us consider the rest of the coordinates in $\tilde{z}$, *i.e.*, the vector $\tilde{z}_2 = \widetilde{V}_\gamma^\top z$. From (3.25) we have $V_\gamma{}^\top = V_\gamma^{-1}$, so

$$\widetilde{V}_\gamma \widetilde{V}_\gamma^\top = I_N - \vartheta_{\mathbf{r}_1}\vartheta_{\mathbf{r}_1}^\top \tag{3.33}$$

and, pre-multiplying $\tilde{z}_2$ by $\widetilde{V}_\gamma$ and using (3.33) we see that $\tilde{z}_2$ equals to zero if and only if $z = \vartheta_{\mathbf{r}_1}\tilde{z}_1$ or, equivalently, if the *synchronisation error* $e \in \mathbb{C}^N$ defined as $e = z - \vartheta_{\mathbf{r}_1}\tilde{z}_1$. equals to zero. That is, $\tilde{z}_2$ constitutes a natural measure of synchrony among the oscillators in the network; it corresponds to the synchrony between each oscillator and the network mean-field.

Thus, the behaviour of the networked systems interconnected via diffusive coupling is naturally and completely captured by the states

$$\begin{aligned} z_m &:= \vartheta_{\mathbf{r}_1}^\top z && \text{(3.34a)} \\ e &:= z - \vartheta_{\mathbf{r}_1} z_m. && \text{(3.34b)} \end{aligned}$$

By using these coordinates we decompose the analysis of the network behaviour in two distinct parts: the first, pertains to the "average" behaviour of the network and the second, to the synchronisation of the units. As we have underlined before, under ideal conditions $z_m \to z_e$ and $e \to 0$. In particular, in the classical consensus paradigm among a balanced network of integrators, the state z_e is constant. For the case of the oscillators (3.14), however, z_e evolves *independently* of the inputs according to what we call *emergent dynamics* that is,

$$\dot{z}_e := \frac{1}{N}\sum_{i=1}^{\mathsf{N}}\big[-|z_i|^2 z_i + \mu_i z_i\big].$$

Moreover, in view of the heterogeneity of the network, one can only expect that the synchronisation errors become arbitrarily small for arbitrarily large values of the interconnection gain γ.

Remark 3 *The vector e corresponds to the errors between each oscillator with state z_i and the scaled and rotated mean-field oscillator, with state z_m. In general, the vector $\vartheta_{\mathbf{r}_1}$ does not necessarily have only rotational components since some of its coefficients are different to one. However, in the limit, as $\gamma \to \infty$, we have $\vartheta_{\mathbf{r}_1} \to \mathbf{1}$ so for sufficiently large values of γ, the elements of $\vartheta_{\mathbf{r}_1}$ converge to $e^{\mathrm{i}\varphi_j}$ where $\varphi_j \in \mathbb{R}$. Thus, for sufficiently large values of γ, the right eigenvector $\vartheta_{\mathbf{r}_1}$ may be considered as a vector of rotations which correspond to the phase difference between the interconnected oscillators and the average oscillator.*

In what follows, we derive the dynamics equations corresponding to z_m and e.

5.2 Dynamics of the averaged oscillator

We differentiate on both sides of (3.34a) and use the network dynamics equation (3.30) to obtain

$$\begin{aligned} \dot{z}_m &= \vartheta_{\mathbf{r}_1}^\top\big[\big(\lambda_1(A_\gamma)I - C(z)\big)z + Dz\big] \\ &= \lambda_1(A_\gamma)z_m - \vartheta_{\mathbf{r}_1}^\top C(z)z + \vartheta_{\mathbf{r}_1}^\top Dz && \text{(3.35)} \end{aligned}$$

however, since $\vartheta_{\mathbf{r}_1}$ is an eigenvector (also) associated to $\lambda_1(D)$, the last term on the right-hand side of (3.35) equals to zero. We proceed to rewrite the rest of the right-hand side of (3.35) in terms of z_m and $\boldsymbol{e}$. From (3.34b), we have $C(\boldsymbol{z})\boldsymbol{z} = C(\boldsymbol{z})[\boldsymbol{e} + \vartheta_{\mathbf{r}_1} z_m]$. Next let us introduce the operator Γ defined as

$$\Gamma(\boldsymbol{z}) := \begin{bmatrix} z_1 & 0 & \dots & 0 \\ 0 & z_2 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & z_N \end{bmatrix};$$

notice that $I_N = \Gamma(\mathbf{1})$, $\Gamma(\boldsymbol{x})\boldsymbol{y} = \Gamma(\boldsymbol{y})\boldsymbol{x}$ and $\Gamma(\boldsymbol{x}) - \Gamma(\boldsymbol{y}) = \Gamma(\boldsymbol{x} - \boldsymbol{y})$ for all $\boldsymbol{x}$, $\boldsymbol{y} \in \mathbb{C}^N$. Also, $C(\boldsymbol{z}) = \Gamma(\boldsymbol{z})^*\Gamma(\boldsymbol{z})$ where Γ^* denotes the conjugate transpose of Γ hence,

$$\begin{aligned} C(\boldsymbol{z})\boldsymbol{z} &= C(\boldsymbol{z})\boldsymbol{e} + \Gamma(\boldsymbol{z})^*\Gamma(\boldsymbol{z})\vartheta_{\mathbf{r}_1} z_m \pm \Gamma(\boldsymbol{z})^*\big[\Gamma(\vartheta_{\mathbf{r}_1} z_m)\vartheta_{\mathbf{r}_1} z_m\big] \\ &= C(\boldsymbol{z})\boldsymbol{e} + \Gamma(\boldsymbol{z})^*\big[\Gamma(\boldsymbol{z} - \vartheta_{\mathbf{r}_1} z_m)\vartheta_{\mathbf{r}_1} z_m + \Gamma(\vartheta_{\mathbf{r}_1} z_m)\vartheta_{\mathbf{r}_1} z_m\big] \\ &= C(\boldsymbol{z})\boldsymbol{e} + \Gamma(\boldsymbol{z})^*\Gamma(\vartheta_{\mathbf{r}_1} z_m)\boldsymbol{e} + \Gamma(\boldsymbol{z})^*\Gamma(\vartheta_{\mathbf{r}_1} z_m)\vartheta_{\mathbf{r}_1} z_m \pm \Gamma(\overline{\vartheta_{\mathbf{r}_1} z_m})\Gamma(\vartheta_{\mathbf{r}_1} z_m)\vartheta_{\mathbf{r}_1} z_m \end{aligned}$$

where $\overline{\vartheta_{\mathbf{r}_1} z_m} = \big[\,\bar{\vartheta}_{\mathbf{r}_{11}}\bar{z}_m \,\cdots\, \bar{\vartheta}_{\mathbf{r}_{1\mathrm{N}}}\bar{z}_m\big]^\top$ and we used $\Gamma(\boldsymbol{z})^* = \Gamma(\bar{\boldsymbol{z}})$. Therefore, using $\bar{\boldsymbol{e}} = \bar{\boldsymbol{z}} - \overline{\vartheta_{\mathbf{r}_1} z_m}$, the linearity of Γ and $|z_m|^2 = \bar{z}_m z_m$, we obtain

$$\begin{aligned} C(\boldsymbol{z})\boldsymbol{z} &= \big[\,C(\boldsymbol{z}) + \Gamma(\boldsymbol{z})^*\Gamma(\vartheta_{\mathbf{r}_1} z_m)\big]\boldsymbol{e} + \Gamma(\bar{\boldsymbol{e}})\Gamma(\vartheta_{\mathbf{r}_1} z_m)\vartheta_{\mathbf{r}_1} z_m + \Gamma(\bar{\vartheta}_{\mathbf{r}_1})\Gamma(\vartheta_{\mathbf{r}_1})\vartheta_{\mathbf{r}_1} |z_m|^2 z_m \\ &= \big[\,C(\boldsymbol{z}) + \Gamma(\boldsymbol{z})^*\Gamma(\vartheta_{\mathbf{r}_1} z_m)\big]\boldsymbol{e} + \Gamma\big(\,[\vartheta_{\mathbf{r}_{11}}^2 \,\cdots\, \vartheta_{\mathbf{r}_{1\mathrm{N}}}^2]^\top\big)(z_m)^2\bar{\boldsymbol{e}} + \Gamma(\bar{\vartheta}_{\mathbf{r}_1})\Gamma(\vartheta_{\mathbf{r}_1})\vartheta_{\mathbf{r}_1} |z_m|^2 z_m. \end{aligned}$$

Using the latter in (3.35), we obtain

$$\dot{z}_m = \lambda_1 z_m - \vartheta_{\mathbf{r}_1}^\top\big[\,C(\boldsymbol{z}) + \Gamma(\boldsymbol{z})^*\Gamma(\vartheta_{\mathbf{r}_1} z_m)\big]\boldsymbol{e} - \vartheta_{\mathbf{r}_1}^\top\Gamma\big(\,[\vartheta_{\mathbf{r}_{11}}^2 \,\cdots\, \vartheta_{\mathbf{r}_{1\mathrm{N}}}^2]^\top\big)(z_m)^2\bar{\boldsymbol{e}} - \alpha\,|z_m|^2 z_m$$

where

$$\alpha = \vartheta_{\mathbf{r}_1}^\top\Gamma(\bar{\vartheta}_{\mathbf{r}_1})\Gamma(\vartheta_{\mathbf{r}_1})\vartheta_{\mathbf{r}_1} \tag{3.36}$$

hence,

$$\dot{z}_m = \big[\lambda_1 - \alpha\,|z_m|^2\big]z_m + f_m(z_m, \boldsymbol{e}) \tag{3.37a}$$

$$f_m(z_m, \boldsymbol{e}) := -\vartheta_{\mathbf{r}_1}^\top\big[\,C(\boldsymbol{z}) + \Gamma(\boldsymbol{z})^*\Gamma(\vartheta_{\mathbf{r}_1} z_m)\big]\boldsymbol{e} - \vartheta_{\mathbf{r}_1}^\top\Gamma\big(\,[\vartheta_{\mathbf{r}_{11}}^2 \,\cdots\, \vartheta_{\mathbf{r}_{1\mathrm{N}}}^2]^\top\big)(z_m)^2\bar{\boldsymbol{e}}. \tag{3.37b}$$

Notice that $f_m \equiv 0$ if $|\boldsymbol{e}|^2 = \bar{\boldsymbol{e}}^\top\boldsymbol{e} = 0$ that is, if synchronisation is achieved asymptotically the dynamics of the average unit, (3.37), converges to the emergent dynamics

$$\dot{z}_e = \big[\lambda_1 - \alpha\,|z_e|^2\big]z_e. \tag{3.38}$$

Hence, a reasonably good measure of stability of the solutions of (3.37a) is that with respect to invariant sets for the solutions of (3.38).

It is also convenient to stress that the interconnection gain γ does not appear explicitly in the right-hand side of (3.37). Nonetheless, it easy to see, from (3.23), that λ_1 is inversely proportional to γ; roughly speaking, $\lambda_1(D) = c + O(\frac{1}{\gamma})$, where the constant c depends only on the matrix $\mathcal{M}$. The same type of relationship is also valid for the eigenvector $\vartheta_{\mathbf{r}_1}$.

5.3 Dynamics of the synchronisation errors

Now we derive the dynamics equation corresponding to the synchronisation error (3.34b). To that end, let us start by introducing the matrix

$$P := \big(I - \vartheta_{\mathbf{r}_1}\vartheta_{\mathbf{r}_1}^\top\big)$$

hence, we have $\boldsymbol{e} = P\boldsymbol{z}$. Next, differentiating on both sides of the latter and using (3.30) we obtain the error dynamics of $\boldsymbol{e}$,

$$\dot{\boldsymbol{e}} = PD\boldsymbol{z} + P\big[\lambda_1(A_\gamma)I - C(\boldsymbol{z})\big]\boldsymbol{z}. \tag{3.39}$$

Now, since $\vartheta_{\mathbf{r}_1}$ is a right eigenvector associated to $\lambda_1(D) = 0$, it follows that $DP = D = PD$. Indeed, we have $DP = D - D\vartheta_{\mathbf{r}_1}\vartheta_{\mathbf{r}_1}^\top$ and $D\vartheta_{\mathbf{r}_1} = \mathbf{0}$ while D and P are both symmetric. Therefore, $PD\boldsymbol{z} = PDP\boldsymbol{z}$ and, since $\boldsymbol{e} = P\boldsymbol{z}$, we obtain $PD\boldsymbol{z} = PD\boldsymbol{e}$. It follows from this and (3.39) that

$$\begin{aligned}\dot{\boldsymbol{e}} &= \big[PD + \lambda_1(A_\gamma)I\big]\boldsymbol{e} - PC(\boldsymbol{z})\boldsymbol{z}.\\ &= \big[D + \lambda_1(A_\gamma)I\big]\boldsymbol{e} - PC(\boldsymbol{e} + \vartheta_{\mathbf{r}_1}z_m)[\boldsymbol{e} + \vartheta_{\mathbf{r}_1}z_m]. \qquad (3.40)\end{aligned}$$

Thus, Equations (3.37) and (3.40) completely define the dynamics of the networked oscillators interconnected via diffusive coupling and in coordinates meaningful for our purposes of analysis. The next section is devoted to the stability analysis of the solutions of these equations, which we regroup for convenience:

$$\dot{z}_m = \big[\lambda_1 - \alpha\,|z_m|^2\big]z_m + f_m(z_m, \boldsymbol{e}), \tag{3.41a}$$

$$\dot{\boldsymbol{e}} = [D + \lambda_1 I]\boldsymbol{e} - PC(\boldsymbol{e} + \vartheta_{\mathbf{r}_1}z_m)[\boldsymbol{e} + \vartheta_{\mathbf{r}_1}z_m]. \tag{3.41b}$$

We investigate two different properties. Firstly, we establish a bound on the synchronisation errors $\boldsymbol{e}$. Then, the second part relates to the stability of the natural attractor of the emergent dynamics (3.38), which corresponds to the nominal part of (3.41a). Notice that this is tantamount to studying the robust stability of an isolated unforced Stuart-Landau equation; more precisely, input-to-state stability with respect to invariant sets of (3.38) and the input $\boldsymbol{e}$. In other words, in the first place, we analyse the behaviour of the solutions of (3.41b) and in the second place, those of (3.41a).

6 Networked systems' stability

6.1 Ultimate boundedness of solutions

As a preliminary but fundamental step in the analysis of Equations (3.41) we formulate conditions that ensure that the trajectories of the networked diffusively-coupled Stuart-Landau oscillators, as described by (3.19) and equivalently by (3.41), are ultimately bounded, see Definition 2.1.The property may be established for Stuart-Landau oscillators, for any interconnection gain $\gamma > 0$, using simple Lyapunov arguments –cf. [69, 61]. For instance, we may invoke the following statement recalled from [47, Theorem 4.18].

Theorem 6 *Consider a system $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$, where $\boldsymbol{x} \in \mathbb{R}^N$ and $f(\cdot)$ is a continuous, locally Lipschitz function. Assume that there exist a $\mathcal{C}^1$ function $V : \mathbb{R}^n \to \mathbb{R}_+$, functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, a continuous positive definite function W and a constant $c > 0$ such that*

$$\alpha_1(|\boldsymbol{x}|) \leqslant V(\boldsymbol{x}) \leqslant \alpha_2(|\boldsymbol{x}|) \tag{3.42}$$
$$\frac{\partial V}{\partial t} f(\boldsymbol{x}) \leqslant -W(\boldsymbol{x}) \qquad \forall \ |\boldsymbol{x}| \geqslant c > 0 \tag{3.43}$$

Then, for every initial state $\boldsymbol{x}(0) = \boldsymbol{x}_\circ \in \mathbb{R}^N$ there exists a constant $T \geqslant 0$ such that

$$|\boldsymbol{x}(t, \boldsymbol{x}_\circ)| \leqslant \alpha_1 \circ \alpha_2(c) \qquad \forall \ t \geqslant T.$$

Then, for Stuart-Landau oscillators, we have the following.

Proposition 3 *Consider the system (3.19), (3.20) and let the graph of the network be undirected and connected. Then, the solutions are globally ultimately bounded and*

$$\begin{aligned} |\boldsymbol{z}(t, \boldsymbol{z}_\circ)| &\leqslant \sqrt{2\bar{\mu}N}, \qquad \forall \ t \geqslant T \\ \bar{\mu} &= \max_{i \in \mathcal{I}} \{\mu_{Ri}, 0\}. \end{aligned} \tag{3.44}$$

Proof. Consider the Lyapunov function candidate $V(\boldsymbol{z}) = \boldsymbol{z}^* \boldsymbol{z}$; it is clear that V satisfies the inequalities in (3.42) with $\alpha_1(s) = \alpha_2(s) = s^2$.

Now, evaluating the total derivative of V along the system's trajectories, using the symmetry of the Laplacian L and the fact that all the eigenvalues of the latter are non-negative, we obtain

$$\dot{V}(\boldsymbol{z}) = \boldsymbol{z}^* F(\boldsymbol{z}) + F(\boldsymbol{z})^* \boldsymbol{z}.$$

where

$$F(\boldsymbol{z}) = -C(\boldsymbol{z})\boldsymbol{z} + \mathcal{M}\boldsymbol{z}, \qquad F(\boldsymbol{z})^* = -\boldsymbol{z}^* C(\boldsymbol{z}) + \boldsymbol{z}^* \mathcal{M}^*.$$

Therefore,

$$\begin{aligned} \dot{V}(\boldsymbol{z}) &= -2\boldsymbol{z}^* C(\boldsymbol{z})\boldsymbol{z} + \boldsymbol{z}^* \big[\mathcal{M} + \mathcal{M}^*\big] \boldsymbol{z} \\ &= -2 \sum_{i=1}^{N} |\boldsymbol{z}_i|^4 + 2\bar{\mu} \sum_{i=1}^{N} |\boldsymbol{z}_i|^2 \\ &= -2 \sum_{i=1}^{N} |\boldsymbol{z}_i|^4 + 2\bar{\mu} \, |\boldsymbol{z}|^2 . \end{aligned} \tag{3.45}$$

On the other hand, notice that

$$\sum_{i=1}^{N} |\boldsymbol{z}_i|^4 \geqslant \frac{1}{N} \, |\boldsymbol{z}|^4 . \tag{3.46}$$

Indeed, we have

$$|\boldsymbol{z}|^4 = \left[\sum_{i=1}^{N} |z_i|^2 \right]^2 = \left[\sum_{i=1}^{N} |z_i|^2 \right] |z_1|^2 + \cdots + \left[\sum_{i=1}^{N} |z_i|^2 \right] |z_N|^2$$

so using the triangle inequality we see that, *for each* $j \leqslant N$,

$$\left[\sum_{i=1}^{N} |z_i|^2\right] |z_j|^2 \leqslant \frac{N}{2}\,|z_j|^4 + \frac{1}{2}\sum_{i=1}^{N} |z_i|^4$$

hence adding up the latter from $j = 1$ to N, we obtain

$$\left[\sum_{i=1}^{N} |z_i|^2\right]^2 \leqslant\ N\sum_{j=1}^{N} |z_j|^4 .$$

Therefore, substituting (3.46) in (3.45) we obtain

$$\begin{aligned}\dot V(\boldsymbol{z}) \ &\leqslant\ -\frac{2}{N}\,|\boldsymbol{z}|^4 + 2\bar\mu\,|\boldsymbol{z}|^2 \\ &= -\frac{1}{N}\,|\boldsymbol{z}|^4 - \frac{1}{N}\Big[\,|\boldsymbol{z}|^2 - 2\bar\mu N\Big]\,|\boldsymbol{z}|^2 .\end{aligned}$$

Thus, from the last inequality, we conclude that $\dot V(\boldsymbol{z}) \leqslant -\frac{1}{N}\,|\boldsymbol{z}|^4$ for all $\boldsymbol{z}$ such that $|\boldsymbol{z}| \geqslant \sqrt{2\bar\mu N}$. It follows, from Theorem 6, that the solutions are globally ultimately bounded and for any $R > 0$ there exists a $T(R)$ such that for all initial conditions such that $|\boldsymbol{z}_\circ| \leqslant R$, the system's trajectories satisfy

$$|\boldsymbol{z}(t, \boldsymbol{z}_\circ)| \leqslant \sqrt{2\bar\mu N} \quad \forall t \geqslant T. \qquad \blacksquare$$

6.2 Practical asymptotic stability of the synchronization errors manifold

In this section we formulate conditions that ensure practical global asymptotic stability of the (not necessarily invariant) set

$$\mathcal{S} = \{\boldsymbol{e} \in \mathbb{C}^N : e_1 = e_2 = \ldots = e_N = 0\}. \tag{3.47}$$

that imply practical synchronisation of the networked Stuart-Landau oscillators. Hence, we show that for large values of the interconnection gain γ the norm of the error $\boldsymbol{e}(t)$ are small and inversely proportional to γ. More precisely, we establish global practical asymptotic stability for the system (3.41b) with respect to the set $\mathcal{S}$. Our analysis relies on Proposition 1 in Chapter 2.

Our main statement in this section is the following.

Theorem 7 *Consider the system (3.19), (3.20) and let Assumption* **A4** *be satisfied. Let* γ^* *be such that* $\Re e[\lambda_2(A_{\gamma*})] \leqslant 0$*. Then, the set* $\mathcal{S}$ *is uniformly globally practically asymptotically stable for all* $\gamma \geqslant \gamma^*$*. Moreover, there exist* $T^* > 0$, c_1, $c_2 > 0$, *independent of* γ, *such that synchronisation errors* $\boldsymbol{e}(t)$ *satisfy*

$$|\boldsymbol{e}(t)|^2 \leqslant \frac{c}{|\,\Re e[\lambda_2(A_{\gamma*})]\,|} \quad \forall\, t \geqslant T^*. \tag{3.48}$$

The previous statement relies mostly upon two properties of the networked system, namely, the negative definiteness of the second smallest eigenvalue of the Laplacian matrix L and uniform boundedness of the trajectories of the network. For a network of the Stuart-Landau oscillators with coupling gain γ it establishes that, for a given arbitrary large ball of initial conditions $B_R = \{\boldsymbol{z}_\circ \in \mathbb{C}^N : |\boldsymbol{z}_\circ| \leqslant R\}$ and an arbitrarily small constant $\delta > 0$, we can always find constants $\gamma(R, \delta)$ and $T^*(R, \delta)$ such that the synchronisation errors $\boldsymbol{e}(t, \boldsymbol{z}_\circ)$ satisfy

$$|\boldsymbol{e}(t, \boldsymbol{z}_\circ)| \leqslant \delta \qquad \text{for all} \ \ t \geqslant T^*.$$

Proof of Theorem 7. Let $\boldsymbol{z}_\circ \in \mathbb{C}$ be initial conditions such that $|\boldsymbol{z}_\circ| \leqslant R$, where the constant $R > 0$ is arbitrary. Let Assumption **A4** generate an orthogonal matrix V_γ and define

$$\boldsymbol{e}_v := V_\gamma^\top \boldsymbol{e}. \tag{3.49}$$

From the latter, (3.41b) and $D = V_\gamma \Lambda_2 V_\gamma^\top$, we have

$$\dot{\boldsymbol{e}}_v = V_\gamma^\top V_\gamma \Lambda_2 V_\gamma^\top \boldsymbol{e} + \lambda_1 \boldsymbol{e}_v - V_\gamma^\top P C(\boldsymbol{z})\boldsymbol{z},$$

which, in view of the orthogonality of V_γ, is equivalent to

$$\dot{\boldsymbol{e}}_v = \Lambda \boldsymbol{e}_v - V_\gamma^\top P C(\boldsymbol{z})\boldsymbol{z}. \tag{3.50}$$

where Λ is defined in (3.28). However, by construction, the first among the N equations in (3.50) is redundant. Indeed, on one hand, we have $\boldsymbol{e}_v = V_\gamma^\top \boldsymbol{z} - V_\gamma^\top \vartheta_{\mathbf{r}_1} z_m$ so, using the identity $V_\gamma^\top \vartheta_{\mathbf{r}_1} = [\,1\,0 \cdots 0\,]^\top$, we obtain

$$\boldsymbol{e}_v = \begin{pmatrix} \vartheta_{\mathbf{r}_1}^\top \boldsymbol{z} \\ \vdots \\ \vartheta_{\mathbf{r}_N}^\top \boldsymbol{z} \end{pmatrix} - \begin{pmatrix} z_m \\ 0 \\ \vdots \\ 0 \end{pmatrix} =: \begin{bmatrix} 0 \\ \tilde{\boldsymbol{e}}_v \end{bmatrix}. \tag{3.51}$$

On the other hand, the first element of $V_\gamma^\top P C(\boldsymbol{z})\boldsymbol{z}$ equals to zero since the first row of $V_\gamma^\top P$ is entirely constituted of zeros. To see this, we observe that

$$V_\gamma^\top P = \begin{bmatrix} \vartheta_{\mathbf{r}_1}^\top \\ \tilde{V}_\gamma^\top \end{bmatrix} \left[I - \vartheta_{\mathbf{r}_1} \vartheta_{\mathbf{r}_1}^\top \right]$$

and recall that, by definition, $\vartheta_{\mathbf{r}_1}^\top \vartheta_{\mathbf{r}_1} = 1$.

Then, let us consider the Lyapunov function candidate $V(\boldsymbol{e}_v) = |\boldsymbol{e}_v|^2$ which is positive definite on the set $\mathcal{S}$. To see this, we refer to (3.51) and observe that $V(\boldsymbol{e}_v)$ is positive definite with respect to the set $\{\tilde{\boldsymbol{e}}_v = 0\}$. Evaluating the total derivative of V along the trajectories of (3.50), we obtain

$$\begin{aligned} \dot{V}(\boldsymbol{e}_v) &= \boldsymbol{e}_v^* \left(\Lambda \boldsymbol{e}_v - V_\gamma^\top P C(\boldsymbol{z})\boldsymbol{z} \right) + \left(\boldsymbol{e}_v^* \Lambda^* - \boldsymbol{z}^* C(\boldsymbol{z}) P^* \overline{V}_\gamma \right) \boldsymbol{e}_v \\ &= \boldsymbol{e}_v^* \big[\Lambda + \Lambda^* \big] \boldsymbol{e}_v + g(\boldsymbol{e}_v, \boldsymbol{z}) \end{aligned}$$

where

$$g(\boldsymbol{e}_v, \boldsymbol{z}) = -\boldsymbol{e}_v^* V_\gamma^\top P C(\boldsymbol{z})\boldsymbol{z} - \boldsymbol{z}^* C(\boldsymbol{z}) P^* \overline{V}_\gamma \boldsymbol{e}_v.$$

Now, since $\boldsymbol{e}_v = [0\ \tilde{\boldsymbol{e}}_v^\top]^\top$ and the first element of $\boldsymbol{z}^* C(\boldsymbol{z}) P^* \overline{V}_\gamma$ equals to zero, we obtain, along the systems' trajectories $\boldsymbol{z}(t)$,

$$\dot{V}(\boldsymbol{e}_v) \leqslant \tilde{\boldsymbol{e}}_v^* \Re e[\lambda_2(A_{\gamma *})] \tilde{\boldsymbol{e}}_v + g(\boldsymbol{e}_v, \boldsymbol{z}(t)) \tag{3.52}$$

where we used the fact that, by convention, $\Re e[\lambda_2(A_\gamma)] \geqslant \Re e[\lambda_i(A_\gamma)]$ for all $i > 2$ and, moreover, by assumption, $0 > \Re e[\lambda_2(A_{\gamma *})] \geqslant \Re e[\lambda_2(A_\gamma)]$ for all $\gamma \geqslant \gamma^*$. In other words, γ^*, which is the lower bound on admissible coupling strength coefficients, is large enough to ensure that the eigenvalues of $-D$ are close enough to those of the Laplacian, hence, non-negative.

Next, we observe that Proposition 3 implies that the solutions of (3.41a) are globally ultimately bounded hence, for any $R > 0$ and any initial conditions such that $|\boldsymbol{z}_\circ| \leqslant R$ there exists a constant $T > 0$ such that

$$|\boldsymbol{z}(t, \boldsymbol{z}_\circ)| \leqslant \sqrt{2\bar{\mu} N} \quad \forall\ t \geqslant T.$$

In turn, it follows from (3.34) and (3.49), that $z_m(t)$ and the synchronisation errors $\boldsymbol{e}(t)$, hence $\boldsymbol{e}_v(t)$, are also uniformly globally ultimately bounded. Moreover, the bound depends only on $\bar{\mu}$ and N. Furthermore, the eigenvalues and eigenvectors of A_γ are uniformly bounded in γ hence, there exists a constant $c > 0$, which depends on $\bar{\mu}$ and N only, such that

$$|g(\boldsymbol{e}_v, \boldsymbol{z}(t))| \leqslant c.$$

From this and (3.52) it follows that

$$\dot{V}(\boldsymbol{e}_v(t)) \leqslant -\left| \Re e[\lambda_2(A_{\gamma *})] \right| |\tilde{\boldsymbol{e}}_v(t)|^2 + c.$$

By direct integration and invoking the comparison theorem, it follows that there exists $T^* > 0$ such that

$$|\tilde{\boldsymbol{e}}_v(t)|^2 \leqslant \frac{c}{\left| \Re e[\lambda_2(A_{\gamma *})] \right|} \quad \forall\, t \geqslant T^*$$

so, from (3.49), (3.51) and the orthogonality of V_γ we obtain (3.48). Global practical asymptotic stability of $\mathcal{S}$ follows from the fact that $\lim\limits_{\gamma^* \to \infty} \Re e[\lambda_2(A_{\gamma *})] = -\infty$.

6.3 Practical asymptotic stability of the invariant set of the averaged oscillator

To complete our analysis, we consider the behaviour of the solutions $z_m(t)$ of (3.41a). Notice that this equation may be regarded as that of a single Stuart-Landau oscillator with a perturbation, that is,

$$\dot{z}_m = (\lambda_1 - \alpha |z_m|^2) z_m + u, \tag{3.53}$$

with $u = f_m(z_m, \boldsymbol{e})$. This equation has exactly the form (3.14), modulo an evident change scale. Therefore, generally speaking, we may use stability theory for perturbed systems with respect to decomposable sets, as discussed in Section 2. Indeed, the origin is an equilibrium point, however, so is the orbit $|z_m| = \sqrt{\lambda_{1\mathsf{R}}/\alpha_{\mathsf{R}}}$, where α is defined in (3.36), which is determined by the complex parameters of the systems in the network, μ_i. More precisely, the set of equilibria is given by

$$\mathcal{W} := \left\{ z \in \mathbb{C} :\ |z| = \sqrt{\frac{\lambda_{1\mathsf{R}}}{\alpha_{\mathsf{R}}}} \right\} \cup \{z = 0\}.$$

Theorem 8 *Consider the network of Stuart-Landau oscillators defined by Equations (3.19), (3.20) and the averaged oscillator of the network defined by (3.34a), whose dynamics is given by equation (3.53). Let Assumption* **A4** *be satisfied. Then, the system (3.53) has the asymptotic gain property and moreover for any $\varepsilon > 0$ there exists a gain $\gamma \geqslant \gamma^*$ such that*

$$\limsup_{t\to+\infty} |z_m(t, z_\circ)|_{\mathcal{W}} \leqslant \varepsilon.$$

Proof. Let $\gamma \geqslant \gamma^*$ and $R > 0$ be arbitrary and consider the system (3.19), (3.20) with initial conditions $\boldsymbol{z}_\circ \in \mathbb{C}$ such that $|\boldsymbol{z}_\circ| \leqslant R$. From Proposition 3 it follows that the solutions of the system (3.19), (3.20) are ultimately bounded hence, there exists a $T = T(R)$ such that (3.44) holds for all $t \geqslant T$.

Now, let us consider the dynamics of the averaged oscillator, (3.53), given by

$$\dot{z}_m = (\lambda_1 - \alpha\,|z_m|^2)z_m + f_m(z_m, \boldsymbol{e}),$$

where $f_m(z_m, \boldsymbol{e})$ is defined in (3.37b). From the latter, we see that $f_m(z_m, \cdot)$ is Lipschitz on compacts of z_m. Moreover, due to the ultimate boundedness of solutions, $z_m(t)$ is uniformly bounded; therefore, there exists a constant $c_3 > 0$ such that, for all $t \geqslant T$, we have

$$|f_m(z_m(t), \boldsymbol{e}(t))| \leqslant c_3\,|\boldsymbol{e}(t)|\,.$$

Thus, invoking Theorem 5 with $u(t) = f_m(z_m(t), \boldsymbol{e}(t))$ and $t \geqslant T$, we see that the solutions of Equation (3.53) satisfy the bound

$$\limsup_{t\to+\infty} |z_m(t, \boldsymbol{z}_\circ)|_{\mathcal{W}} \leqslant \eta(\|\boldsymbol{e}\|_\infty).$$

Furthermore, from Theorem 7, there exist constants $T^* > T$ and $c > 0$, independent of γ, such that for all $t \geqslant T^*$, the synchronisation errors $\boldsymbol{e}(t)$ satisfy (3.48). It follows that

$$\limsup_{t\to+\infty} |\,z_m(t, \boldsymbol{z}_\circ)\,|_{\mathcal{W}} \leqslant \eta\left(\left[\frac{c}{|\,\Re e[\lambda_2(A_{\gamma^*})]\,|}\right]^{1/2}\right) \qquad \forall\, t \geqslant T^*$$

and, by repeating the same argument as before, $i.e.$, observing that

$$\lim_{\gamma^*\to\infty} \Re e[\lambda_2(A_{\gamma^*})] = -\infty,$$

and using the fact that $\eta \in \mathcal{K}_\infty$, we obtain that, for any $\varepsilon > 0$, there exists a $\gamma > \gamma^*$ such that

$$\eta\left(\left[\frac{c}{|\,\Re e[\lambda_2(A_{\gamma^*})]\,|}\right]^{1/2}\right) \leqslant \varepsilon.$$

We conclude that

$$\limsup_{t\to+\infty} |\,z_m(t, \boldsymbol{z}_\circ)\,|_{\mathcal{W}} \leqslant \varepsilon.$$

That is, the invariant set $\mathcal{W}$ is practically asymptotically stable, in the sense that, by increasing the interconnection gain γ, we can make solutions $z_m(t, \boldsymbol{z}_\circ)$ converge arbitrarily close to $\mathcal{W}$. ■

7 Example: network of four Stuart-Landau oscillators

For the sake of illustration, we consider a network of four diffusively coupled Stuart-Landau oscillators. The graph of interconnections is assumed to be undirected and its structure is depicted in Figure 3.2.

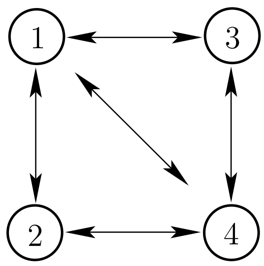


Figure 3.2: Symmetrically connected graph with four nodes

The dynamics of the interconnected oscillators is given by

$$\dot{z}_j = \big[-|z_j|^2 + \mu_{\mathsf{R}j} + \mathrm{j}\omega_j \big] z_j + \gamma \sum_{i=1}^{N} d_{ji}(z_i - z_j) \quad j \in \{1 \dots 4\} \tag{3.54}$$

with $\omega_1 = 5$, $\omega_2 = 12$, $\omega_3 = 18$, $\omega_4 = 7$. We assume, for simplicity, that the constants $\mu_{\mathsf{R}i}$ are all equal to $\mu_{\mathsf{R}} = 4$. The Laplacian matrix L defined by the interconnection coefficients $d_{ij} = 1$, for all i, j, is given by

$$L = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}.$$

We performed some numerical simulations of this network in SIMULINK™ of MATLAB™ with different values of the interconnection gain γ. For this purpose we re-write the system's equations (3.54) in polar coordinates, with $z_i = r_i e^{\mathrm{i}\theta_i}$, $\dot{\theta}_i = \omega_i$. Hence, we have

$$\begin{aligned} \dot{r}_j &= [\mu_{\mathsf{R}} - r_j^2] r_j + \gamma \sum_{i=1}^{N} d_{ji} \big[r_i \cos(\theta_i - \theta_j) - r_j \big], \qquad j \in \{1 \dots 4\}, \\ \dot{\theta}_j &= \mu_{\mathsf{I}j} r_j + \gamma \sum_{i=1}^{N} d_{ji} r_i \sin(\theta_i - \theta_j). \end{aligned}$$

The synchronous limit cycle is generated by the emergent dynamics equation,

$$\dot{z}_m = \big[\lambda_1 - \alpha\, |z_m|^2 \big] z_m \tag{3.55}$$

with $\alpha = \vartheta_{\mathbf{r}_1}^{\top} \Gamma(\overline{\vartheta}_{\mathbf{r}_1}) \Gamma(\vartheta_{\mathbf{r}_1}) \vartheta_{\mathbf{r}_1}$ –see (3.36). The synchronisation frequency and the amplitude of the synchronous limit cycle is completely defined by the first eigenvalue λ_1 of $A_\gamma = \alpha I - \gamma L + \mathrm{i}\Omega$, with $\Omega =$diag$[\omega_i]$ and its corresponding eigenvectors. Then, the radius of the limit cycle is given by $R = \sqrt{\lambda_{1\mathsf{R}} / \alpha_{\mathsf{R}}}$ and the average frequency of oscillation is $\omega_m = \mathcal{I}mg(\lambda_1)$.

The phase portraits of the system against time are depicted in Figure 3.3 (for the case $\gamma = 0$), Figure 3.4 (for the case $\gamma = 4$) and Figure 3.5 (with $\gamma = 10$). It may be appreciated that each oscillator rapidly reaches a common limit cycle when the coupling strength is sufficiently large. Moreover, it is clear that their oscillating frequencies converge to a common value as γ increases. The latter is yet clearer from the plot in Figure 3.6, which represents the evolution of the oscillating frequencies $\omega_i = \dot{\theta}_i$ with $\gamma = 10$.

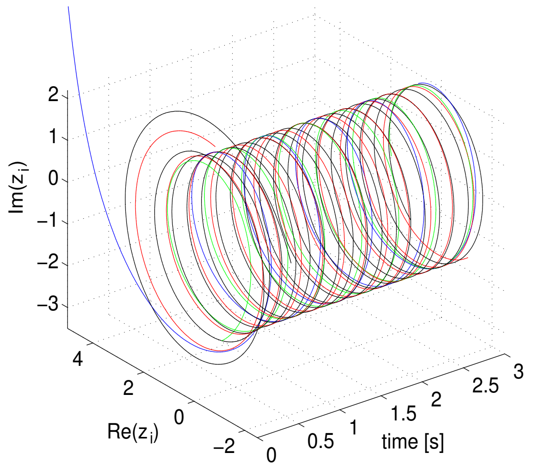


Figure 3.3: Phase portrait of the four limit cycles, against time, with the four oscillators being decoupled, *i.e.*, with $\gamma = 0$.

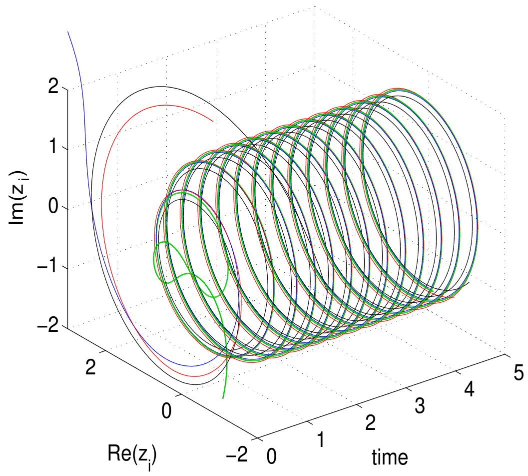


Figure 3.4: Phase portrait of the four limit cycles, against time, with coupling strength $\gamma = 4$.

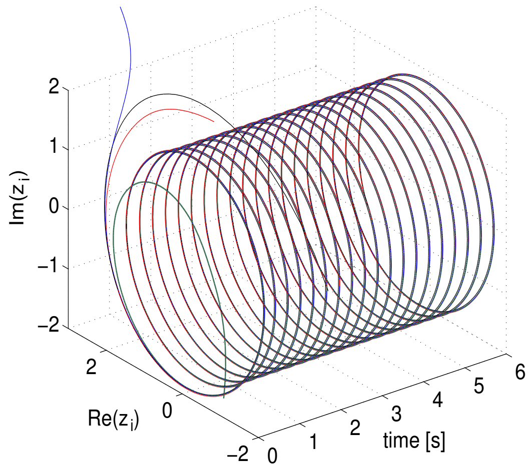


Figure 3.5: Phase portrait of the four limit cycles, against time, with coupling strength $\gamma = 10$.

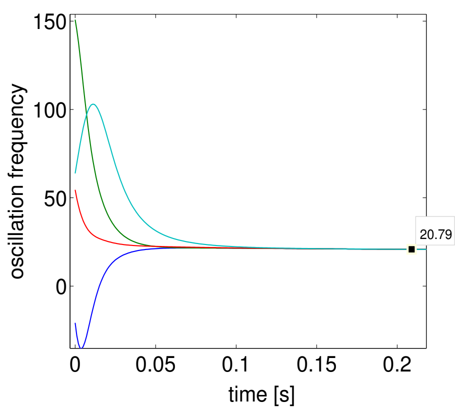


Figure 3.6: Evolution of oscillating frequencies with coupling strength $\gamma = 10$.

CHAPTER 4

Conclusions and future research

As we said in the Introduction, there exist two competing orthogonal trends of thought, reductionism and emergionism. The results that we presented in this document do not allow to (in)validate one or the other, even for the particularly simple systems that we considered in the previous chapter. Nevertheless, our results suggest that while it may be accepted that the network model may be reduced, *asymptotically* (for infinitely strong interconnections) to a weighted average, while for intermediate values of interconnection gains a collective behaviour emerges from the systems' interactions. A clarification of this and other fascinating questions may only come with further study.

We wrap up this memoir with a brief discussion on research areas which would benefit of a general framework for analysis and design of networked systems, from a control-theory perspective. In particular, we describe a few of the research directions that we have discovered after the work that we have performed in the area in the last few years, notably, through the supervision of the two PhD thesis [27, 18]. These research perspectives stem from the framework that we have initiated and briefly described in the previous chapters.

1 Specific research domains on networked systems

1.1 Neuroscience

Brain-related sciences (from neuro-physiology to psycho-physics) aim at building models of perception, action, coordination, conscience, context awareness, adaptation, and decision making; an enormous amount of facts and findings concerning neural behaviours has been collected. Yet, there is a considerable lack of models and interpretations of these phenomena. Adequate models of neuronal activity and of the propagation of electrical stimuli produced by electrodes, as well as corresponding simulation tools, must be developed. Based on these models new (closed-loop) stimulation techniques may be considered, making stimulation methods more adapted to human physiology and more individualised.

Notions such as *state awareness*, *state-dependent decision making*, *identification by anticipa-*

tion, which appear in the relevant literature, show a striking similarity to system theoretic concepts of state reconstruction, feedback, *etc*. Although the level of complexity of systems in brain sciences may be intimidating, this similarity is much more than a simple resemblance of language –these problems are actually the same.

In the image of generic research on networked systems, the impact of that in the realm of neuronal networks transcends disciplines and cannot be overestimated. A clear example is that of *deep-brain stimulation*. Since its invention in the early 90's, this treatment has become a standard symptomatic treatment of Parkinson's disease, it consists in permanently stimulating deep brain zones such as basal ganglia, through implanted electrodes. Since Parkinsonian symptoms are known to be linked to coherent neuronal hyper-activity in basal ganglia, synchronisation and stability analysis may play a key role in determining the underlying process of the onset of these pathological oscillations and the role played by somatotopy in this synchronisation.

The analysis of synchronisation properties of more realistic models of neuronal networks, taking into account aspects such as *plasticity* and *non linearity of couplings*, noise and delays is also fundamental. Yet, in spite of the successful *practice*, the development of the founding theory is only at its *debut*, not only in what modelling concerns but, consequently, control design and stability analysis for open and closed-loop electrical deep brain stimulation. The following are particular but significant open questions that may be addressed from our control-theory perspective:

- To understand the effects of intrinsic and input/output connections of the basal-ganglia nuclei on the synchronisation properties of small populations of neurons; analysis of the conditions for synchronisation in a somatotopic organisation. Control theory tools may help to characterise the patterns of neuronal activity in basal ganglia nuclei in healthy and Parkinson-attained individuals.
- To characterise and to analyse patterns of neuronal activity in basal-ganglia nuclei for healthy and Parkinsonian cases. This includes, among other issues, the characterisation of changes in firing-rate dynamics and neuronal synchrony resulting from local and network level synaptic interactions.
- To design controllers for deep brain stimulation and establish formal stability analyses of population level models of the basal ganglia under effects of electrical stimulation.

At the same time, there is clear evidence that medical applications can benefit of system and identification theory both in the development and design of so-called population models, that is, macroscopic-level models of neuronal networks. Indeed, this is a domain where systems and control methodologies and tools are not yet of standard use but certainly will be in a near future –*cf*. [30].

1.2 Power-systems' networks

Electric power networks are among the largest and most complex man-made systems and exhibit very complicated behaviours. As a result, there are many badly understood phenomena caused by the interaction of such a large number of devices and the large spatial dimensions. Currently, major changes of the structure of the grid are being implemented, in particular to support the large-scale introduction of renewable energy sources such as wind farms and solar plants. Two crucial features of these sources of electric power should be addressed: most renewable sources are dispersed over a wide geographical area, and most of them fluctuate largely over time. In addition, power networks are affected by different economic, social and environmental factors which vary over time but at lower scale, relative to that of the dynamics of power plants and electric grids. An important area of research is how to handle the imperfections of the models used in all layers, as well as the lack of knowledge on disturbances and on external influences as, $e.g.$, future prices, demands and capacities. Consequently, such systems may, and should, be described by multi-scale models.

Control of power networks should be structured hierarchically in layers and organised in such a way that subsystems have local controllers and exchange information among each other or with a central coordination mechanism. These local controllers have only limited information about the evolution of other subsystems and the hierarchical structure of the system. The adaptation of methods and tools from self-organised critical systems theory, and emergent behaviour, to analysis of multi-scale networks is required to ensure the robustness of distributed systems to failures of components or communication links.

1.3 Social systems' networks

It is generally accepted that social and economy pertain to complex systems and both deterministic and stochastic descriptions are needed to define the main features of their dynamics. During the last decade, social networks have been attracting substantial attention within the research community. In particular, a tremendous opportunity exists to bring in quantitative tools to analyse the dynamics of such networks as well as to study the impact of such networks on decision making.

In these networks the nodes represent the individuals while the interconnections represent a type of relationship among individuals. We note that these interconnections may have positive or negative signs. Although actual social learning can involve very complicated dynamics, some effects may be captured in terms of characteristics of signed graphs and of the approximate dynamics of individual decision-making.

To date, the results in this direction rely on idealised situations, $e.g.$, in which all agents have the same utility and where there is no disruption. Furthermore, they only address asymptotic learning and fail to analyse the influence of external effects (such as media, injecting outside agents, changing the network topology). These external stimuli can clearly be recasted as *inputs* or *perturbations* to the network average behaviour.

2 Directions of research – dynamical systems perspective

We have identified a handful of aspects which, while being fundamental to each specific domain, naturally lead, through mathematical abstraction, to common problems of analysis and design. We sustain that control theory is *the* discipline to address these paradigms, via a general framework of analysis of networked systems built upon the study of synchronisation and emergent network behaviour.

From the problems described above it may now be clearer that, as we have remarked in the Introduction, thefollowing aspects play a fundamental role in the analysis and control of networked systems:

- the coupling strength;
- the network topology;
- the type of coupling between the nodes, $i.e.$, how the units are interconnected;
- the dynamics of the individual units.

Even though the technical results presented in the previous chapters address some of them in a systematic way, they remain preliminary in view of the restrictive assumptions that we impose to the system. At the same time, they also lead to other fascinating open problems on control and stability theory of networked systems; below, we describe a few.

- **Network topology.** The framework for which we have laid the basis must be extended to deal with networks whose graph is not necessarily undirected and connected. Besides the case of directed graphs we shall consider *signed* graphs, $i.e.$, in which links with both positive and negative weights of the interconnections may appear. This type of interconnections corresponds to neuronal networks with both inhibitory and exhibitory connections amongst the neurons or neuronal populations, or signed interconnections in social networks.

Furthermore, one must consider disconnected graphs. In particular, questions related with clustering (roughly speaking, a graph cluster is a connected component of a graph) must be addressed and the effects of clustering in weakly connected networks or multi-layered networks must be considered. In other words, more general models of network interconnections must be taken into account.

- **Modelling network interconnections.** Much of the current research on networked control systems is focused on fixed structures for communication and interaction. Certainly, due to its relative simplicity, the majority of existing results on control and stability of networked systems, deal with the case of static linear interconnections and, at best, time-varying. However, in many applications, agents are interconnected in a more complex manner $e.g.$, nonlinear, hybrid or, even, *dynamic* –see [84]. For instance, networks of mobile vehicles or smart energy grids with time-varying sets of producers and consumers give rise to system structures that change over time.

The restrictive hypothesis of linearity must be relaxed to account for interconnections modelled as nonlinear functions of the states of the network units (which corresponds $e.g.$, to the simple model of synaptic connectivity in the neuronal models) and, in a second step, by assuming that the interconnections may have their own dynamics (which corresponds to modelling neuronal plasticity). Furthermore, delays and signal degradation are common especially when the distance between elements of a communication system is considerable.

In addition, control methods for networked systems in a broad framework must be able to cope with the fact that communication links are (or can be) established only in a limited period of time. Ensuring stability and performance for systems in which communication or coupling between subsystems is restricted in this way goes far beyond the present investigations of time-delays or (single) packet losses in communication. The methods of stability and stabilisation under conditions of persistency of excitation that we have developed –see [67, 57], seem well fitted for such scenarios.

- **Time-scale separation.** After our experience on analysis of networks of nonlinear oscillators –see [19, 20, 21, 22, 30, 29, 28], we know that it is possible to view a network as the interconnection of two systems: the emergent dynamics and the synchronisation error system. Besides the advantages that we have already underlined, another interest of this approach is that it allows one to analyse the network as a system in two time-scales: one related to the dynamics of the average node and one to the network itself.

This analysis framework was developed for the case of fully-interconnected networks. Nevertheless, many networks, as for instance social, biological and neuronal, often present a natural partition of communities which may be described via the notion of *graph clustering*, known in network theory. Extending the scope of our framework to cope with several-time-scales networks will make it possible to include in the model multiple effects induced by a multilayered structure of the system ($e.g.$, as in electrical networks), hierarchical network structure and, possibly, dynamics of network interconnections. Refining our results obtained so far on practical stability to the case of networks with multiple time scales would allow to analyse cluster networks, in particular, robustness with respect to perturbations of the graph structure.

- **Robustness to perturbations.** The complex systems, as those described in the previous section, are "open" systems since they never function in isolation but in interaction with an environment. Therefore, analysis of such systems must also take into account uncertainties and perturbations induced by this interaction. Perturbations in network topology, computation, or communication may propagate errors throughout the network that can degrade performance or, worse, result in positive feedback loops which tend to amplify the effect of the errors, thereby destabilising the system. Another important aspect is the analysis of robustness with respect to misbehaviour of agents, due to either accidental faults or malignant intrusion of potential adversaries. Distributed control, with aim at establishing a separation principle, imposes itself as a possible solution.

- **Network separation principle.** The separation principle from classical control theory establishes, for the systems that admit it, that the dynamics of the overall system may be separated in dynamics of two or several subsystems. Although the separation principle is typically associated to observer-based control, the notion is extendable, for instance, by using our theory on cascaded systems, to other scenarios of complex interconnected systems –see [56]. Such an approach naturally leads to simpler methods of analysis and design. This is especially significant in order to establish approximate results in the area of synchronisation of networked systems.

Furthermore, a separation principle inevitably brings us to study the systems' interconnections. For instance, while a failure in a single node of a networked system can typically be tolerated better than in a centralised system, distributed systems are potentially vulnerable to domino effects.

Closing remarks

The paradigm of (control of) networked systems is extremely broad, its multiple facets concern a variety of networks. These may be technology-based as in telecommunications and energy distribution or they may appear naturally, as in the case of ecosystems, social networks and biological systems at cellular level. Abstracting an application problem to solve it via sound mathematical approaches is a fundamental and distinctive characteristic of control theory. If, moreover, the abstraction captures essential features of a class of problems then the results obtained in one given application domain can be transferred to other domains which, *prima facie*, appear completely different. For twenty-five years, this has been our approach to scientific research[1] ; the technical results presented here are no exception. Nevertheless, these constitute only a first step in the research path that they open.

[1]–see "Summary of developed research topics" on p. 91.

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PARTIE II– Dossier de carrière scientifique

Curriculum vitae

Résumé des thèmes de recherche développés

Elena Panteley

CURRICULUM VITAE

Chargé de recherche 1re classe au CNRS

1 Etat Civil

Date de naissance :	28 Décembre 1963
Nationalité :	Russe
Statut actuel :	Chargé de recherche 1re classe au CNRS
Etablissement/labo :	Laboratoire des signaux et systèmes, UMR CNRS 8506
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2 Titres Universitaires

1997– Doctorat en Mathématiques Appliquées,
Université d'Etat de St. Petersbourg,
Faculté de Mathématiques et Mécanique,
Intitulé: "Algorithmes de commande pour des systèmes électromécaniques et systèmes mécaniques à contraintes"
Directeur : Dr. S. V. Gusev.

1986– Master en Mathématiques Appliquées,
Université d'Etat de St. Petersbourg,
Faculté de Mathématiques et Mécanique.
Intitulé : "Commande de systèmes de locomotion bipèdes dans la phase de double support"
Directeur : Dr. S. V. Gusev.

3 Parcours

1er Octobre 2004-présent : Chargé de recherche au CNRS 1re classe
Division Systèmes, Laboratoire des Signaux et Systèmes, UMR 8506
Supélec, Gif sur Yvette

Janvier 1999 à janvier 2000 : Chercheur Post-doctorant
Institut National de Recherche en Informatique et Automatique, Rhône Alpes.

- Projet BIP. Superviseur : Prof. Bernard Espiau.
  Activités : Commande optimale du robot BIP.
  Encadrement de DEA.
  Analyse et commande des systèmes non linéaires.

Février à décembre 1998 : Chercheur Associé, *Center for Control Engineering and computation* (CCEC), University of California, Santa Barabara, CA, USA.

Superviseurs : Prof. Petar Kokotović and Prof. Andrew Teel
Activités : Analyse et commande des systèmes non linéaires.
Commande des systèmes non holonomes.
Commande adaptative.

1986-1998 : Institut pour Problèmes de Génie Mécanique,
Académie des Sciences de Russie.

Poste : Chercheur permanent.

[1990-1998]: Au laboratoire " Control of Complex Systems ".
Activités : Analyse de stabilité des systèmes.
Commande adaptative non linéaire des systèmes mécaniques.
Robots manipulateurs à contraintes holonomes.
Commande de systèmes électromécaniques.

[1986-1990]: Au Laboratoire " Large Scale Control Systems "
Activités : Modélisation et commande des systèmes de locomotion bipèdes.

4 Activités d'enseignement

Cours de 3^{e}cycle

- **Automnes 2009–2013:** Cours " Comportement des solutions de systèmes dynamiques " (18h). Master de Recherche 2^{e}année, *Information, Systèmes et Technologies*, spécialité " Automatique et traitement du signal et des images", Université Paris-Sud.
- **Juillet 2011:** Cours "Controlled synchronisation of dynamical systems" (5h). Dispensés à des étudiants de master et doctorat à l'Université Nationale Autonome du Mexique, UNAM, Mexico.

- **Février 2011:** Cours de doctorat " Controlled Synchronisation of Dynamical Systems " (21h). Dispensés (avec A. Loria) à des doctorants dans le cadre de la " Graduate School of European Embedded Control Institute ", Gif-sur-Yvette.
- **Avril 2010:** Cours de doctorat " Integral Conditions for Uniform Asymptotic Stability of Dynamical Systems " (5h). Dispensés à Yildiz Technical University, Istambul
- **Janvier 2010:** Intervention dans le cours "Controlled synchronization of dynamical systems" (1h), Graduate School of European Embedded Control Institute, Gif-sur-Yvette.
- **Octobre 2009** "On Matrosov theorem for time-varying systems" (6h). Cours de doctorat dispensés (avec A. Loria) à Norwegian University of Science and Technology en coopération avec le Centre for Ships and Ocean Structures (CESOS).
- **Automne 2007:** Cours " Comportement des solutions de systèmes dynamiques (7h), Master de Recherche 2eannée, *Information, Systèmes et Technologies*, spécialité "Automatique et traitement du signal et des images", Université Paris-Sud.
- **Avril 2006:** Cours de doctorat " Stability and stabilisation of time-varying systems ". Dispensés dans le cadre de la Formation en Automatique à Paris, subventionée par l'UE à travers l'action Marie Curie " Control Training Site ".
- **Avril 2005:** Cours de doctorat " Stability of Nonlinear Cascaded systems " (6h). Dispensés (avec A. Loria) dans le cadre du " Strategic University Program (SUP) on Computational Methods in Nonlinear Motion Control (CMinMC) "en collaboration avec le " Centre for Ships and Ocean Structures (CESOS) at the Norwegian University of Science and Technology (NTNU) ".
- **Mars 2005:** Cours de doctorat "Tools for analysis and control of time-varying systems" (9h). Dispensés dans le cadre de la Formation en Automatique à Paris, subventionée par l'UE à travers l'action Marie Curie " Control Training Site ".

Cours de 2ecycle

- **Septembre 1989 - Mai 1990:** " Conception de lois de commande à l'aide a d'ordinateur " (160h), St. Petersbourg, Russie.

5 Activités liées à l'administration

- **2007-présent** (Co-)responsable des cours en automatique pour le Master de Recherche 2eannée, *Information, Systèmes et Technologies*, spécialité "Automatique et traitement du signal et des images", Université Paris-Sud.
- **2012:** Membre externe de la commission de spécialistes pour le recrutement d'un MC 61, profil Automatique, à l'Université de Caen, printemps 2012.
- **2006-2010** Membre de la Commission de spécialistes de l'Université Paris-Sud, Section 61.

6 Activités liées à la recherche

6.1 Prix reçus pour un article

2010: Best Student Paper Award in the area of Signal Processing Systems Modeling and Control (ISYNCO 2010), pour *E. I. Grotli, A. Chaillet, E. Panteley, and J. T. Gravdahl, " Robustness of ISS systems to inputs with limited moving average, with application to spacecraft formations"*

1992: 1er Prix à la " Baltic Olympiade on Automatic Control for Postgraduate Students".

6.2 Participation à des comités, organisation de conférences, workshops

Comités de spécialistes

- **2012–présent:** Membre du "IFAC Committee on Nonlinear systems "
- **2012:** Membre externe de la commission de spécialistes pour le recrutement d'un MC 61, profil Automatique, à l'Université de Caen, printemps 2012.
- **2010–présent:** Membre du comité exécutif du Réseau d'excellence HYCON2
- **2008:** Rapporteur pour la Commission Européenne pendant le " EU-Russia Information and Brokerage Event ", Moscou, novembre 2008.

Participation à des jurys de thèse

- **2013:** Rapporteur. Chong, Michelle Siu Tze, Parameter and state estimation of nonlinear systems with applications in neuroscience, The University of Melbourne, 2013 ;
- **2010:** Examinateur. Ioannis Sarras, Sur la conception constructive des lois de commande et d'observateurs pour des systèmes mécaniques via passivité, immersion et invariance, 8 avril 2010 ;
- **2009:** 2009: Examinateur. Fernando Castanos, Cyclo passivité et commande par interconnexion, Supélec, LSS (Université Paris Sud), 3 sept. 2009 ;
- **2007:** Examinateur. Nicolas Guenard, Optimisation et implémentation de lois de commande embarquées pour la téléopération de micro drones aériens X4-flyer, Université de Nice-Sophia Antipolis, 29 octobre 2007 ;
- **2006:** Présidente du jury, Alessandro de Rinaldis, Sur la compensation des ondes de reflexion, Supélec, LSS (Université Paris Sud), 3 avril 2006 ;
- **2004:** Examinateur. Paloma Moya, Commande adaptative des systèmes non linéaires non linéairement paramétrisés : Application aux systèmes de réaction, Supélec, LSS (Université Paris Sud), 2004.

Organisation et animation de colloques

- **2015:** Membre du comité de programme international du "IFAC Conference on Modelling, Identification and Control of Nonlinear Systems", juin 2015.
- **2014:**
  - Membre du comité d'organisation du workshop "Computational Biology, Computational Neuroscience and Modern Control Theory: on the Path to Symbiosis", LSS, Supelec, 30 Juin - 1 Juillet 2014.
  - Membre du comité de programme international du " European Control Conference ECC 2014 ", Juillet 2014.
- **2013:**
  - Membre du comité de programme international du "9th IFAC Symposium on Nonlinear Control Systems ", septembre, 2013
  - Membre du comité d'organisation du " Interdisciplinary Symposium on Signals and Systems for Medical applications", Paris, juin 2013.
  - Co-organisator of the Workshop HYCON2-AD3 on Biological and Medical systems, Paris, 2013.
- **2011:** Co-Chair du "Control Systems Technical Programm Commitee" du "3rd International Congress on Ultra Modern Telecommunications and Control Systems", Budapest, Hongrie, 2011.
- **2010:** Co-Chair du "Control Systems Technical Programm Commitee" du "2nd International Congress on Ultra Modern Telecommunications and Control Systems", Moscou, Russie, 2010.
- **2009:** Publicity chair pour le comité d'organisation du " IEEE Multi conference on Systems and Control", Saint Petersbourg, Russie, 8-10 juillet 2009.
- **2009:** Membre du comité d'organisation du " Networked embedded and control system technologies: European and Russian R&D cooperation " organisé avec la conférence ICINCO à Milan, Italie, juillet 2009. Cadre : NESTER, PCRD 7, EC.
- **2008:** Co-organisateur et animatrice de la session "Embedded systems computing and control session", within the framework of the EU-Russia FP7 ICT conference organised by the European Commission on the 21rd and 22nd October 2008 in Moscow.
- **2007:** Membre du comité de programme international du " IFAC Workshop Adaptation and Learning in Control and signal processing (ALCOSP'07)", St Petersbourg, Russie, 29-31 août 2007,
- **2006:** Membre du comité d'organisation pour le " CTS-HYCON Workshop on Nonlinear and Hybrid Control, Paris", 10-12 juillet 2006.

- **2000:** Co-animatrice de la Journée-école : Analyse et Commande Nonlinéaire, Centre Michel-Ange, Paris, juin 2000.

6.3 Programmes d'échanges, collaborations, réseaux internationaux, projets nationaux et Européens

Projets et contrats

- **2014** : Projet iCODE Workshop " Computational Biology, Computational Neuroscience and Modern Control Theory: on the Path to Symbiosis", 2014.

- **2010-2014** : " *FP7 Network of Excellence Hybrid Control: Taming Heterogeneity and Complexity of Networked Embedded Systems* (HYCON2)",
  Membre du comité exécutif de HYCON2 et co-porteur du package *Application Domain " Biological and Medical applications*.

- **2010-2011** : PEPS-INSIS project TREMBATIC " Automatique et Stimulation électrique : vers l'obtention de lois de commande réalistes pour l'atténuation du tremblement pathologique ". Participants : L2S, LIRMM-Montpellier, CHU Montpellier et le Centre de réducation fonctionnelle Propara. L'objectif du projet est la modélisation et la conception de lois de commande pour la stimulation électrique, notament la stimulation fonctionnelle et la stimulation cérébrale profonde.

- **2008-2009** FP7 CSA project Networked embedded and control systems technologies for Europe and Russia (NESTER) http://www.nester-ru.eu/. Coordinatrice scientifique et porteuse du contrat. Projet porté sur l'identification de priorités de collaboration, et l'établissement de cette dernière, entre la Russie et l'Europe dans le domaine des systèmes de commande embarqués.

- **2008** Projet "Synchronisation et Commande pour des Applications de Gestion d'Energie" soutenu par l'Ambassade de France en Russie. Co-porteuse du projet.

- **2007-2010** Project PICS avec la Russie entitulé "Robust and adaptive control of complex systems." Participants : CNRS (LAAS et LSS) ainsi que l'INRIA pour la France et l'Académie de Sciences de Russie ("Institute for Problems of Mechanical Engineering" (IMPE-RAS), " Institute for Control Sciences" (ICS-RAS) et " Nizhny-Novgorod State Technical University"

- **2004-2008** Membre du Réseau d'Excellence HYCON (Hybrid Control: Taming Heterogeneity and Complexity of Networked Embedded Systems), Commission Européenne PCRD 6.

- **2002-2006** Participation au " Marie Curie Control Training Site". J'ai été membre du comité local d'organisation du " CTS-HYCON Workshop on Nonlinear and Hybrid Control " organisé à Paris en 2006. J'ai, également, co-rédigé deux livres de notes de cours issus de la Formation en Automatique de Paris.

- **2000-2003** Co-auteur du projet de collaboration NSF-CNRS " Analysis and synthesis of Nonlinear time-varying control systems".

- **1999-2001** Participation au projet "Robot à pattes" dans le cadre du "GdR Automatique du CNRS", avec le support du MENRT.
- **2000-2001** Participation au projet "De nouveaux outils pour l'analyse des systèmes variants dans le temps : Application à la commande" dans le cadre du "GdR Automatique du CNRS ".
- **1997** Co-auteur du projet de recherche pour bourse post-doctorale avec le Conseil National de la Recherche Scientifique Norvegien (NFR) pour deux ans.
- **1994-1998** Participation à un projet de collaboration INTAS avec les pays NIS) ex-URSS). Coordonnateurs Dr. R. Ortega (HEUDIASYC URA CNRS 817, Compi'egne, France) and Prof. A. L. Fradkov (IPME, Acad. of Sc. of Rusia).

Mobilité et visibilié

Conférencier invité

- **2014** Workshop "Model reduction across disciplines", Août 2014, University of Leicester, Royaume Uni.
- **2013** Symposium "Interdisciplinary Symposium on Signals and Systems for Medical Applications", (ISSSMA), Paris, 2013.
- **2011** CLaSS Concertation Meeting : Contribution of Control to Complex Systems Engineering, Bruxelles, June 2011 : Presentation together with E. Camacho of the HYCON2 Position Paper on Systems and Control in FP8 : Contribution of systems and control science to the challenges of future engineering systems (co-authored by HYCON2 NoE Leaders)
- **2009** Présentation à la Session invitée au " IEEE Multi-conference on Systems and Control ", St Petersbourg, Russie.
- **2006** Workshop on Group Coordination and Cooperative Control, Tromsø, Norway.
- **2000** Session : "New Lyapunov tools for stability and stabilisation of nonlinear systems" on International Symposium on Mathematical Theory of Networks and Systems, Perpignan, France.
- **1999** Workshop "New directions in nonlinear observer design", 24-26 June 1999, Geiranger Fjörd, Norway.

Chercheur invité

- University of Leicester, Royaume Uni (1 semaine), Mars 2014
- CINVESTAV, Mexico, Mexique (1 semaine), Avril 2014
- Universidad Nacional Autónoma de México, Mexique (1 semaine), Avril 2014

- CRAN (Nancy) : Groupe Commande et Observation des Systèmes, collaboration scientifique sur la modélisation et la commande des neurones, (2 jours, séminaires), 2012.
- UNAM (Mexico), Electrical Engineering Section of the Graduate School of Engineering (DEPFI), préparation d'un projet PICS et collaboration scientifique; (3 semaines), 2011.
- UNAM (Mexico), Electrical Engineering Section of the Graduate School of Engineering (DEPFI), collaboration scientifique, seminaires, 2009;
- Yildis Technical University (Istanboul, Turquie), Dept. of Control and Automation Engineering, April 2010 (1 semaine) ;
- Institute for Problems of Mechanical Engineering, Russian Academy of Sciences, visite dans le cadre d'un projet PICS. Présentation " Neuronal stimulation: control theory perspective " au " St. Petersbourg Control Seminar ". Septembre 2010
- Division of Systems and Control, Dept of Information Technology, Uppsala University, Seminar "Deep brain stimulation - control theory perspective", Nov. 2010 (3 jours)
- Equipe Bio-Electromagnetisme, IMS, Bordeaux (un jour), janvier 2009, seminaire.
- Université Nationale Autonome du Mexique (UNAM), Dept. Ingénierie, Mexico, Mexique (une semaine) Novembre 2003.
  Invitée par Prof. Gerardo Espinosa.
- CICESE, Ensenada, Mexique (une semaine). Novembre 2003.
  Invitée par Prof. Rafael Kelly.
- CCEC, University of California at Santa Barbara, (2 semaines). Décembre 2001.
  Invitée par Prof. Andrew Teel.
- CICESE, Ensenada, Mexique (une semaine). Juin 1998.
  Invitée par Prof. R. Kelly.
- Norwegian University of Science and Technology, Department of Engineering Cybernetics, Norvège. Novembre-Decembre 1997 (6 semaines).
  Invitée par Prof. Olav Egeland, dir. du département Génie Cybernétique.
- Norwegian University of Science and Technology, Department of Engineering Cybernetics, Norvège. Août 1997 (4 semaines).
  Invitée par Prof. Olav Egeland, dir. du département Génie Cybernétique.
- University of Twente, Enschede, Pays Bas. Mars 1997 (2 semaines).
  Invitée par Prof. Henk Nijmeijer (actuellement à l'Univ. de Eindhoven).
- University of Delft, Delft, Pays Bas. Mars 1997 (2 jours).
  Seminaire : "Cascaded control of rigid robots with induction motor drives".
  Invitée par : Prof. Jacquelien Scherpen.
- Université Catholique de Louvain-La-Neuve, Louvain-La-Neuve, Belgique.
  Novembre 1996 (2 jours).
  Invitée par : Dr. Rudolphe Sepulchre.

- Université de Technologie de Compiègne, Compiègne, FRANCE,
  Novembre 1996 (3 semaines).
  Invitée par : Dr. Romeo Ortega (actuellement au LSS).

- Université de Technologie de Compiègne. Janvier-Février 1995 (5 semaines).
  Invitée par : Dr. Romeo Ortega.

- Linkoping University, Linkoping, Sweden, Septembre 1993 (une semaine).
  Seminaire : "Adaptive control scheme for constrained robots".
  Invitée par Prof. Lennart Ljung.

6.4 Actions de valorisation

- **2013** " HYCON 2 Position Paper on Systems in Control ". Réf.[2] [122]. Dans le cadre de la préparation du PCRD 8 de l'Union Europénne à l'horizon 2020.

- **1997** Ford Motor Co., Detroit MI, Etats-Unis, contrat portant sur la commande d'un moteur turbo-diesel.

6.5 Activités d'administration liées à la recherche

- **2011-2013** Animation de l'équipe "Syncro" dans la Division Systèmes du LSS ; équipe composée de 2 DR1-CNRS, 2 CR1-CNRS, 2 MdC Paris-Sud ainsi qu'un nombre variable de chercheurs temporaires (post-doctoraux et doctorants).

- **2010-2014 FP7 Network of Excellence HYCON2** : Actuellemnt, je suis

  - membre du Comité Exécutif de HYCON2,
  - coordinateur du *Application Domain* "**Biological and Medical Applications**"; en particulier, responsable du domaine "Medical Applications".
    Ce dernier inclut toutes les activités de HYCON 2 (modélisation et commande) en rapport avec la stimulation électrique.

  Parmi mes activités, se trouvent :

  - la préparation du "HYCON2 Position Paper on Systems and Control" pour le PCRD 8 : "Contribution of systems and control science to the challenges of future engineering systems"
  - la présentation de ce dernier (avec E. Camacho) pendant le *CLaSS Concertation Meeting* : "Contribution of Control to Complex Systems Engineering" à Bruxelles, juin 2011.
  - en tant que leader du *Application Domain* "Biological and Medical Applications":
    - la préparation du texte de projet HYCON2,

[2]Pour les citations, se référer ̀la *Liste exhaustive de publications*.

  - la description des deliverables,
  - la préparation des deliverables en rapport avec "BIO/MED applications", D.2.4.5, D.1.5.1, D.9.3.1, D.8.4.2.
  - l'organisation du workshop "BIO/MED ", juin 2013.

- **2009-présent:** Co-organisateur de l'EECI Graduate School on Control, Supelec, http://www.eeci-institute.eu. Saisons 2009–2013, 18 à 21 modules dans 4 pays différents, 300 étudiants/an.

- **2008-2009:** Projet *Coordinated Support Action* **NESTER**

  - Coordinatrice d'un des quatre *Work packages*
  - Responsable de la préparation de deux deliverables:
    "Interview's report" et "Report on 4 selected industrial sectors".

  D'autres activités incluent :

  - ma participation à la négotiation de budget avec la CE;
  - la préparation du "kick-off meeting", avec participation;
  - la préparation et animation de la session "Embedded Systems Computing and Control" organisée par le programme ICT de l'Union Européenne et le projet NESTER lors du " EU-Russia Information and Brokerage Event", Mouscou, Russie, 2008;
  - ma participation au comité d'organisation du " 2nd workshop linked to ICINCO conference ", à Milan, Italie, juillet 2009;
  - la préparation du rapport final;
  - présentation des rapports périodiquess et du rapport final.

  NB: Le projet NESTER a été conclu avec succès en octobre 2009 ; notamment, tous les délivrables ont été acceptés –voir le rapport de la CE en annexe.

7 Encadrement de jeunes chercheurs

7.1 Post-doctorants

- **I. Haidar** , "Analysis of synchronization in basal ganglia neuronal populations" avril 2012'avril 2013 (co-encadré avec W. Pasillas-Lepine et A. Chaillet). Son travail de recherche porte sur l'analyse de synchronisation neuronale dans le ganglion basal au niveau mésoscopique. Une attention particulière a été portée sur l'hétérogénéité des interconnexions et les retards. Publications : [3], [46], [51]. Poste actuel : Post-doc, L2S.

- **D. Efimov** Post-doc LSS, Oct. 2006- Dec. 2007, bourse Post-doctoral du Ministère de la Recherche (1 year) and HYCON post-doc scolarship (2 months). Publications : [67], [68], [69], [71], [72], [73], [74]. Poste actuel : Chargé de recherche, INRIA.

7.2 Thèses de doctorat

- **A. El Ati** (100%) "Commande de neurones dans la maladie de Parkinson en utilisant un modelage au niveau macroscopique". Univ. Paris Sud, LSS, 2010-2013, bourse du Ministère de la Recherche. Soutenance : Décembre 2014.

  Publications : [44], [48], [50], [55], [47].

- **L. Conteville** (100%) "Analyse de la stabilité des réseaux d'oscillateurs non-linéaires, applications aux populations neuronales".

  Univ. Paris-Sud, LSS, 2008-2013. Soutenue, le 17 oct 2013

  Avec bourse fléchée du Ministère de la Recherche,

  Publications : [49], [52], [53], [54]

- **R. Postoyan** (33%) Commande et construction d'observateurs pour des systèmes nonlinéaires incertains à données échantillonnées et en réseau.

  Soutenue le 26 nov 2009 à l'Univ. Paris Sud, LSS,

  (co-encadrée avec F. Lamnabhi-Lagarrigue et T. Ahmed-Ali).

  Publications : [63]

- **S. Theodoulis** (50%) avec with Gilles Duc, SUPELEC, oct. 2006-dec. 2008, bourse BDI de la fondation EADS.

- **E. Kyrkjebø** , Norwegian University of Science and Technology (NTNU), stage de 8 mois en visite au LSS dans le cadre du réseau Marie Curie CTS, mars-décembre 2005. Il s'agit d'un réel encadrément bien que non officialisé par aucun accord inter-universitaire. Publications : [35]

7.3 Master, DEA

- **2013:**

  - S. Sakhraoui, " Analysis of diffusively coupled nonlinear oscillators: Control theoretic approach ", stage de M2R, Université Paris Sud, encadrement à 100%, printemps 2013.
  - Huu-Tam Pham, stage de M2R (avec Gilney Damm et Philippe Egrot) " Evaluation of maximum share of renewable energy sources in a synchronous region and the effect of energy storage on its stability ".

- **2011:** D. Zonetti "An Hamiltonian approach to power system modeling and analysis", stage de M2R, Université Paris Sud, co-encadrement à 50%, printemps 2011.

- **2010:** A. El Ati, Comportement neuronal dans la maladie de Parkinson: modelisation et analyse au niveau macroscopique, stage de M2R, Université Paris Sud, co-encadrement à 100%, printemps 2010.

- **2008:** J. Tessieras, TER M1 - IST, projet de conception SUPELEC (avec A. CHAILLET), Analyse et simulation des phénomènes de synchronisation neuronale, printemps 2008.
- **1999:** C. Azevedo "Trajectory generation for biped locomotion" , co-encadrement à 90%, INRIA Rhône Alpes, printemps 1999.

7.4 Stages ingénieur

- **2012:**
  - S. Azouvi, projet de conception, SUPELEC (avec A. CHAILLET), "Etude de la cohérence neuronale par une approche entrée-sortie", printemps 2012 ;
  - A. Mestivier, projet de conception SUPELEC (avec A. CHAILLET), binôme avec S. Azouvi ;
- **2011:**
  - A. Catherin, projet de conception SUPELEC (avec A. CHAILLET), Validation d'un modèle d'évolution du taux de décharges neuronales dans les zones cérébrales impliquées dans la maladie de Parkinson, printemps 2011 ;
  - A. Gauduel, projet de conception SUPELEC (avec A. CHAILLET), binôme avec A. Catherin ;
- **2010:**
  - B. Amoudruz, projet de conception SUPELEC, Supélec, (avec A. CHAILLET), "Sur la propriété de robustesse du modèle de Kuramoto et sa linéarisation", printemps 2010.
  - T. Charpentier, projet de conception SUPELEC, Supélec, (avec A. CHAILLET), binôme avec B. Amoudruz, printemps 2010.
- **2008:** S. Zouiten, stage 2e année Supélec, (avec A. CHAILLET), Modèle Simulink et interface graphique pour la simulation d'un réseau de cellules neuronales, 2 mois, 2008.

Summary of developed research topics

To summarise my research work carried out over about 25 years I present here a brief account of the main topics in which our results have had an appreciable impact in the scientific community, as well as in my close entourage. As it is schematically represented in Figure 4.1, my experience lays firmly on control theory, seen as a discipline of mathematics applied to engineering; notably, this is the case of several technological problems solved by efficient solutions via theoretical tools. The accompanying detailed curriculum vitae as well as the exhaustive list of publications constitute factual proof of my interest for multidisciplinary research. Indeed, an approach hand in hand between applied science and theoretical developments has led me to study automatic control in a variety of areas ranging from stability of dynamical systems to engineering disciplines such as robotics, as well as nonlinear physics and neuroscience.

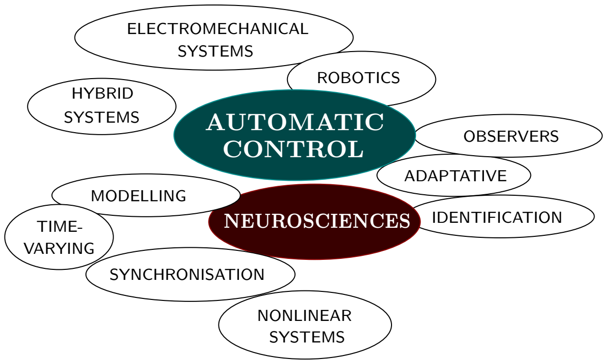

Figure 4.1: Developed research topics

At the beginning of my career I worked on modelling and control of biped robots (1990s) and later, I carried out pioneering work on control of robot manipulators under holonomic constraints (1997). After my PhD thesis (1997)[3] I adopted as main subject of research, that of control design and stability analysis of time-varying systems, topic in which I involved A. Loria (now DR2-CNRS) from the end of his PhD in 1996. The work that we accomplished became a reference in the field.

The beginning of my career within CNRS (mid-2000s) focused on the study of hybrid systems, notably, through the supervision of post-doctoral researcher D. Efimov (now CR-INRIA). Later, driven by scientific curiosity and attraction for fundamental research, I adopted the study of neuroscience and nonlinear physics, notably, the analysis and synchronisation of oscillators. A few years ago, I introduced these research topics to the *Laboratoire des signaux et systèmes* thereby constituting an informal research team in which I involved younger researchers and lecturers (W. Pasillas, CR1-CNRS, A. Chaillet, MdC Univ P. Sud). A byproduct of our collaboration was the common supervision of I. Haidar (post-doc) and the basis for research grants led by A. Chaillet.

[3]Even though I defended my PhD in 1997, I started working as a permanent "Chargé de recherche" for the Academy of Sciences, since 1989.

In what follows, I present a brief summary of research topics that I have developed with a number of collaborators and particular control problems that we solved with supervised PhD students and post-docs. The summary is succinct but representative of the work that we have accomplished so far. The topics are not presented in a strict chronological order but they are reminiscent of our research-direction transitions. The articles cited correspond to the exhaustive list of publications on p. 107.

1 Control of mechanical and electro-mechanical systems

Our activities on control of mechanical and electro-mechanical systems mostly date back to the first years of our career (1990s) however, this is a topic on which we have worked constantly as it is often a valuable source of inspiration for more general studies on dynamical systems. Below, we mention some of the contributions that we made in this area that fascinated the community in previous decades.

- **Biped locomotion and constrained manipulators.** At a first stage we were concerned with the problem of control of biped locomotion in a sugital plane with particular attention to the double-support phase. During the latter, the biped's motion is restricted by holonomic constraints that lead to a situation in which part of the generalised coordinates are uniquely defined in function of the remaining independent coordinates. Based on this fact, in [117] we proposed a reduced-order dynamic model and, based on this model, we designed diverse controllers for the trajectory control problem of the biped during the double-support phase [113, 119].

- **Constrained robots control.** The reduced order model proposed in [117] was extended to the case of manipulators interacting with an infinitely stiff environment, in the sense that an equation for the dependent coordinates was also included in the model, in order to control also the reaction force. The main characteristic of this model is that, under some suitable assumptions, the constrained manipulator model can be *decoupled*; this decomposition naturally yields a *cascaded* structure of two subsystems –see Figure Figure 4.2 on p. 95. The implication of this is that one can design *separately* controllers to steer the generalised trajectories and the constraint forces to the desired goal references. See [21, 27, 106, 107, 108].

Using the reduced-order model we also proposed a nonlinear observer and proved for the first time, asymptotic stability of the closed loop system considering an infinitely stiff environment. It is worth mentioning that in [106] we addressed the practically important problem of output feedback control of constrained manipulators with bounded controls.

- **Adaptive friction compensation for global tracking of robot manipulators.** In [25] we treated the practically interesting problem of global tracking of manipulators in the presence of friction, assuming that friction is represented by the dynamic model. We considered the case when only robot positions and velocities are measured and all the system parameters (robot and friction model) are unknown.

- **Control of certain nonholonomic systems.** In collaboration with Erjen Lefeber and Henk Nijmeijer, formerly with the Univ. of Twente, we addressed the problem of designing simple global tracking controllers for a kinematic model of a mobile robot and a simple dynamic model of a mobile robot. For this we used a cascaded systems approach, resulting into *linear* controllers that yield K-exponential stability. As a consequence we showed that the positioning and the orientation of the mobile car can be controlled independently of each other. The preliminary work [99] served as "launching platform" for the PhD thesis of E. Lefeber (2000).

- **Control of a turbo-diesel engine.** In collaboration with Dr. A. Sokolov (formerly at Ford Motor Co., Detroit) we considered the problem of desired set-point stabilisation for a turbo-charged diesel engine using a simplified 3-dimensional model. Even though the engine model is highly nonlinear, the use of a cascaded systems approach (see farther below) allowed to construct a simple *linear* controller which guarantees global asymptotic stabilisation of the desired operating point. This work, part of which was published in [22], was carried out within a consultancy contract for Ford Motor Co. See our Curriculum Vitae on p. 87.

- **Robustness under passivity-based control.** With our colleague R. Ortega we analysed robustness properties of mechanical systems controlled by the well-known Passivity-based Control technique of Interconnection and Damping Assignment (IDA-PBC), [36, 82]. The IDA-PBC approach, developed by R. Ortega (LSS) and co-authors, which consists in the design of a "reshaped" Hamiltonian function for the system (analog of a Lyapunov function) that allows to conclude asymptotic stability of the closed loop system but has only semi-negative derivative. For open-loop stable systems we analysed how sign-indefinite damping injection can asymptotically stabilise the system and we proposed a constructive procedure to reduce the problem to the solution of a set of partial differential equations.

Continuing the same line of research we considered the problem of *asymptotic* stabilisation of nonlinear, "double integrator", open–loop stable systems via sign–indefinite damping injection, see [59]. A constructive procedure to reduce the problem to the solution of a set of *partial differential equations* was presented. Particular emphasis was given to mechanical systems, for which it was shown that the proposed approach obviates the usual detectability assumption needed to conclude asymptotic stability via LaSalle's invariance principle.

In collaboration with Henk Nijmeijer we addressed the problem of robust stabilisation of nonlinear systems affected by time-varying uniformly bounded affine perturbations. Our approach relies on the combination of sliding mode techniques and passivity-based control. Roughly speaking we show that under suitable conditions the sliding-mode variable can be chosen as the output of the perturbed system in question. Then, we showed how to construct a controller which guarantees the global uniform convergence of the plant's outputs towards a time-varying desired reference, even in the presence of permanently exciting time-varying disturbances. As applications of our results we addressed the problems of tracking control of the van der Pol oscillator and ship dynamic positioning. See [98].

Our interest for tracking control problems and, specifically, the control of electromechanical systems via a separation principle, naturally led us to study nonlinear time-varying systems in a fairly general setting.

2 Stability and stabilisation of dynamical systems

Generally speaking, since the late 1990s we have been studying necessary and sufficient conditions for stability of dynamical systems in the sense of Lyapunov, more particularly, uniform asymptotic stability. Our contributions range over a large class of systems: continuous-time and sampled-data, differential equations or differential inclusions (discontinuous right-hand sides), stability of compact sets (such as a point in the space - most usual) and unbounded sets. Besides linearisation, methods of analysis of Lyapunov stability for nonlinear systems may be roughly classified in three categories: differential methods, difference methods and integral methods. The second method of Lyapunov which consists in finding a Lyapunov function with a negative definite derivative belongs to the first class as well as converse theorems or statements in the spirit of invariance principles (Barbashin, Matrosov, La Salle). Difference-equations-based methods aim to relax regularity assumptions on the Lyapunov function and demand that, evaluated along the system's trajectories, it satisfies a difference comparison equation. The third class is somewhat dual to the first as it requires that the trajectories satisfy certain integrability properties along the system's trajectories.

We have widely contributed to the three methods with generalisations of popular tools such as invariance principles and the so-called Barbalat's lemma. Regarding the former, we introduced an extended Matrosov's theorem which allows to study more general classes of time-varying systems than the classical Matrosov's theorem –see [12, 88, 124]. On the other hand, our research on integral conditions extend Barbalat's lemma by allowing to establish uniform stability hence, robustness with respect to bounded disturbances –see [16, 93, 96]. More significantly, our integral conditions serve to establish the extended Matrosov's theorem and many other powerful theoretical tools such as for cascaded time-varying systems –see [18, 24, 26, 37, 77, 80, 99, 100, 101, 103, 104, 105].

Furthermore, as mentioned earlier, our results apply to different classes of systems and guarantee different types of stability. For instance, in [62] we established a new characterisation of exponential stability for nonlinear systems which involves Lyapunov functions that may be upper and lower bounded by any monotonic functions satisfying a growth order relationship rather than being just a state's norm in some degree. In particular, one may allow for Lyapunov functions with arbitrary weakly homogeneous bounds. Furthermore, it is convenient to mention that, most commonly, stability is formulated as a property of an equilibrium, however, in some applications and, in particular, in the problems linked with practical stability or synchronisation, these properties are formulated in terms of closed sets –see [10, 70, 77, 78, 79, 8, 16].

2.1 Cascaded nonlinear systems

Cascaded systems are an important class of systems in stability and control theory since they arise naturally in control design for many engineering applications. The fundamental problem in analysis of cascaded systems is to establish under which condition(s) two stable systems remain stable after being coupled in a cascade configuration –see Figure 4.2. In the realm of linear systems it is well established that the so-called separation principle holds that is, the poles of two linear systems interconnected in cascade remain independent of each other hence, the stability and performance of one has no effect on the other's. For instance, this principle is fundamental to design output-feedback controllers which make use of state observers; indeed, according to the

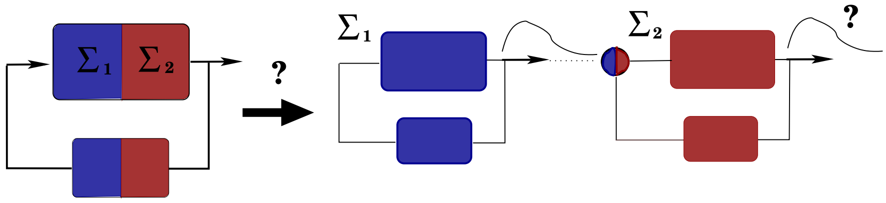

Figure 4.2: A feedback-interconnected system may be considered as a cascaded system, for the purpose of analysis

certainty- equivalence principle stability is conserved if one implements a stabilising state-feedback controller by replacing the unmeasured states by estimates provided by a convergent estimator. Yet, in general, one may expect that two stable nonlinear systems become unstable when interconnected in cascade.

The separation principle is also useful in the opposite sense, as an analysis tool. In this regard, the fundamental question is to determine under which conditions one may consider a (complex) system as the cascade interconnection of subsystems. This problem is naturally motivated by the control of many engineering systems; a clear example is that of electromechanical systems which, at least ideally, may be considered as the cascade interconnection of an electrical and a mechanical system.

Motivated by problems on tracking control, I was first interested on non-autonomous systems. Since our preliminary results, for Euler-Lagrange systems published in [26], we have developed a framework of study which consists in a number of validated theoretical tools that extend the separation principle beyond the realm of linear time-invariant systems to that of nonlinear time-varying systems [105, 103, 22, 18, 24]. In [24] we established sufficient conditions in the form of an integrability property of the “perturbing” state[4]. In [18] we identified three classes of complementary systems, in accordance with growth-rate conditions on the dynamics of the perturbed system.

Thus, our contributions in the field are off-the-shelf theorems that may be easily applied to establish stability and often serve as guidelines to design simple-to-implement, engineering-intuitive control laws. Some applications of our results include the following:

- tracking control of mobile robots (see, *e.g.*, [99]);
- tracking control of under-actuated marine vehicles (see, *e.g.*, [20]);
- control of robot manipulators with AC drives (see, *e.g.*, [26]);
- diesel engines (see, *e.g.*, [22]);
- formation control of vehicles (see, *e.g.*, [6, 33, 61]);
- a separation principle for Euler-Lagrange systems (see, *e.g.*, [38]).

[4]The conditions in this paper were later reformulated in terms of integral input to state stability in Arcak *et al*, “A unifying integral ISS framework for stability of nonlinear cascades, SIAM J. Control and Optimization, 2002, vol. 40, pp. 888-1904.

2.2 Stability of sets

The formalism on stability with respect to sets pertains to the study of the solutions of a system relatively to a configuration rather than an operating point, which is the most common case, *i.e.*, that of stability of an equilibrium[5]. For instance, one may be interested in the study of stability of a biped robot relative to a periodic orbit in the phase space, in the convergence of trajectories towards a neighbourhood of an operating point or in the stability and attractivity properties of a manifold. Hence, in view of its broad scope, the formalism of stability with respect to sets may be utilised to approach a variety of analysis problems for diverse types of dynamical systems, in a unified and systematic manner.

Our work in this area concerns the large class of systems described by set-valued maps and differential inclusions; for instance, discontinuous systems. We have developed new characterisations of stability and convergence in the form of an integrability property of the trajectories –see [96, 93, 16, 19]. Discrete-time couter-parts of the latter were published in [8]. The conditions that we establish consist in boundedness of the integrals of nonlinear functions (positive definite) of the solutions of the differential inclusion. These conditions are related to the $\mathcal{L}_p$ stability concepts, well-known in theory of signals, and are equivalent to the more common conditions from Lyapunov theory, expressed in differential form. Thus, this approach allows to establish *necessary and sufficient* conditions for the asymptotic and uniform exponential stability with respect to sets for a very broad class of systems (which includes non-autonomous systems).

Another advantage of the integral criteria is that they make it easy to establish stability properties in the input-output sense. This follows from the interconnection properties and auxiliary functions (*e.g.*, storage) as it is the case of dissipative systems, stable in the $\mathcal{L}_2$ sense.

Moreover, these results have led us to a generalisation of the so-called Matrosov's theorem and the establishment of characterisations of uniform asymptotic stability of nonlinear systems under output injection [12]. These results are widely used in practical applications since they typically allow considerable simplifications.

2.3 Persistency-of-excitation as a tool for system analysis and control

Persistency of excitation is a concept that was introduced about 45 years ago in the context of system's identification of unknown constant parameters. Loosely speaking, the system is made to operate in a way that all modes are excited and, then, based on state measurements, one can design identification laws which roughly consist in a gradient search for the values of the parameters. It has been showed that for a class of linear systems the regressor having this property is necessary and sufficient for the convergence of estimation errors to zero.

For many years, abundant theory was developed for adaptive *linear* control systems and it has been abusively, if not incorrectly, used to study *nonlinear* adaptive control systems. Typically, the gradient identification law depends on a complex multivariable function of time and states, called regressor and which may loose rank, causing the estimation to be stagnated. We developed a

[5]Note that even the problem of tracking control, that is, stabilisation with respect to a trajectory, may be recasted in a problem of set-point stabilisation, that of a non-autonomous system.

solid theory for nonlinear systems upon a new concept of persistency of excitation, tailored for nonlinear functions. We extended the concept of persistency of excitation to the case of functions depending on the time and state of the system under consideration. This notion allowed us to formulate new results on stability of nonlinear time-varying systems in the case when the derivative of the Lyapunov function is negative *semi*definite. Our stability theorems and definitions, take into account this state dependence of the regressor and guarantee *uniform* global asymptotic stability.

Our main results have filled in important theoretical gaps in adaptive control [19, 94] and corrected long-standing mistakes present in related literature –$cf.$ [95]. Particularly, we have established proofs of uniform asymptotic stability for nonlinear systems under so-called model-reference adaptive control [123, 87]. Beyond identification and adaptive control, our research on stability analysis based on persistency of excitation has also led to the solutions of open problems for systems which may not be stabilised via smooth time-invariant feedback. For instance, we have made significant contributions in the area of stabilisation and tracking control of nonholonomic systems via time-varying feedback; controllers with the property of persistency of excitation –see [99, 14, 88, 91, 97].

2.4 Adaptive control

Besides our work on persistency of excitation, which has a direct impact in adaptive control by correcting long-standing mistakes in related literature, we have worked on more specific problems, mainly with R. Ortega (DR1-CNRS) from L2S.

- **Adaptive control in robotics applications.** We treated the practically interesting problem of global tracking of manipulators in the presence of friction, assuming that friction is represented by the dynamic model. We considered the case when only robot positions and velocities are measured and all the system parameters (robot and friction model) are unknown. See [25].

Also, several adaptive control algorithms were proposed for position/force control of robots interacting with infinitely stiff environment for the cases of unknown robot and/or surface parameters [27, 111, 107].

- **Application of integral tools.** In [15], we were interested in global adaptive stabilisation of nonlinear systems in the case that know a linearly parametrised stabiliser and a Lyapunov function for the ideal closed-loop system such that the standard estimator is implementable, but this Lyapunov function is not strict, $i.e.$, its derivative is only negative semi-definite. Our motivation to consider this situation stems from the fact that, for many physical systems, the natural Lyapunov function candidate is the total energy, which is never strict. To complete the stability proof in this case it is necessary to add a –rather restrictive– detectability assumption. Our main contribution is to show that it is possible to overcome this obstacle by adding to the parameter estimator an identification error coming from an indirect identifier. We thus replace the detectability assumption by a new condition that essentially requires that the negative-definite terms appearing in the Lyapunov function derivative dominate the uncertain terms that cannot be matched by the controller.

- **$\mathcal{L}_1$ adaptive control.** Roughly speaking $\mathcal{L}_1$ adaptive control, which appeared in the years 2006–2011, suggests to replace the tracking error-driven parameter estimator of direct model reference adaptive controller with a state prediction-driven one, used in indirect schemes. In our papers [132, 4, 39] we showed that the architecture of $\mathcal{L}_1$-adaptive control is, indeed, different from classical model reference adaptive control and (after some transient) it asymptotically coincides with a simple full-state feedback, linear time-invariant proportional integral controller.

As the next step in the same direction, in [40] we investigated when a parametrised controller, designed for a plant depending on unknown parameters, admits a realisation which is independent of the parameters. We remark that adaptation is unnecessary for this class of parametrised controllers.

3 Hybrid systems: stability and stabilisation

Hybrid systems, which were at the core of my research project based on which I was hired as *chargé de recherche* by CNRS, in 2004, pertain to the case in which the models of complex interconnected systems, $i.e.$, system of systems, are composed of equations of different mathematical nature, such as: discrete-time, continuous-time, algebraic relations, partial differential equations, difference and differential inclusions, and even non-mathematical models but *ad hoc* look-up tables.

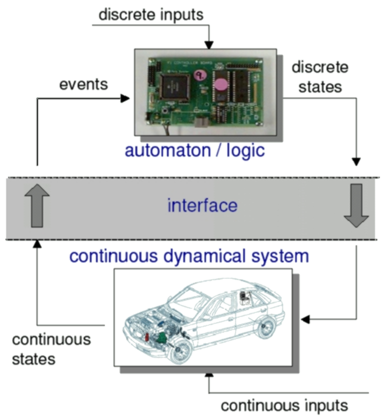

Figure 4.3: Schematical representation of an example of a hybrid system

Based on our experience on nonlinear, especially time-varying, systems, we have developed a number of theoretical tools but we have also addressed several problems in the realm of hybrid systems. Two of the main topics that we have developed in this area are sampled-data systems and switched systems.

3.1 Sampled-data systems

We have extended our results on stability of sets for differential inclusions to discrete-time and sampled-data systems. In particular, we studied necessary and sufficient conditions for (global) uniform asymptotic stability for systems of parametrised difference inclusions. These conditions involve summability criteria on trajectories of the system to conclude global asymptotic stability and represent discrete-time counterparts of our previous results on integral stability criteria for differential equations and inclusions. These summability criteria are tailored to be used for stabilisation of sampled-data systems via their approximate discrete-time models and allow to conclude semi-global practical asymptotic stability, or global exponential stability, of the sampled-data system via appropriate properties of its approximate discrete-time model.

In the context of stabilisation of sampled data systems we considered the problem of stabilisation of arbitrary (not necessarily compact) closed sets for sampled-data nonlinear differential inclusions and, in particular, stabilisation of arbitrary closed sets, plants described as sampled-data differential inclusions and arbitrary dynamic controllers in the form of difference inclusions. See [8, 10, 70, 78]

3.2 Analysis and design of switched systems

Switched systems are a special kind of hybrid systems which are characterised by a switching law that determines which, among a set of models, defines the solution of the dynamical system over a period of time, called *dwell-time*. The switching law, as well as the dynamic models, may be of many kinds hence, to fix the ideas, we focus on a set of continuous-time models defined by ordinary differential equations and a switching law which takes the form of a piecewise-constant function taking only positive integer values. The latter correspond to indexes with which the models are tagged.

In such scenario, an interesting and well-studied problem is that of determining under which conditions on the switching function the solution trajectories of the switched system remain stable, provided that each system is stable separately. Correspondingly, the control design problem may consist in designing a switching control law to "choose" one among a bank of stabilising controllers to drive a plant for "some time". This control approach is known as supervisory control and it was introduced by S. Morse in the 1990s.

There exist many good reasons and practical motivations to use a set of controllers for a single plant as opposed to one controller : for instance, we may think of a complex system whose dynamic behaviour can only be described satisfactorily by using several models, each corresponding to a mode of the system or, in other words, each model being valid for state values in a specified region of the state space. Furthermore, a common situation in control practice is that of models which consist in look-up tables based filled-up with experimental data. Then, it is common practice to implement a group of "local" controllers applied depending on the operating mode. One more common scenario in which supervisory control is useful is that in which design constraints are imposed by certain "(sub)optimality" goals : for instance, one may ask for a controlled plant to track an operating point under constraints regarding transient performance, speed of convergence (to the desired operating point), robustness with respect to uncertainties or measurement noise, $etc.$ The term sub-optimality is, therefore, understood in that sense : to obtain, for instance, the fastest speed of convergence among a set of possibilities (not necessarily all possibilities).

While theoretically challenging in general, focused problems where switching between a local and a global controller brings solutions otherwise difficult or even impossible to achieve are common in the control of mechanical systems. On a more general basis, switching amid more than two controllers imposes significant challenges to analysis since classic stability theory does not apply. For instance, as it is well-known, switching between stable modes of a closed-loop control system does not necessarily yield a stable behaviour. Over the past years there has been an exponentially increasing interest on stability and stabilisation of switched systems, notably for linear switched systems. An important trend is that based on Lyapunov-like methods as for instance, finding a common Lyapunov function (–$cf.$ J. Dafouz and co-workers in Nancy, France) invariance principles; geometric methods, $etc.$, to mention a few.

In our work, we have studied stability of switched systems and supervisory control in a general setting in which stability and performance are measured with respect to an output, that is, mathematically speaking, a function of the state: possibly a variable with a physical meaning but which does not necessarily correspond to the state itself. Given several nonlinear systems with certain input-output characterisation we formulated sufficient conditions that ensure that the switching system with dwell time or average dwell time possesses the same input-output properties.

Then, these results served as a background for several supervisor-based control design schemes for the nonlinear systems via uniting several controllers for nonlinear systems. The stabilisation goal is to bring an output motion close to a desired operating point. Moreover, it appears natural to consider robustness aspects hence to study stability in an input-output sense. This is the context in which we place our main results in this area. Furthermore, we are interested in stability for systems with inputs that is, our results in this area serve to draw conclusions about the robustness of closed-loop systems with respect to external perturbations.

See [7, 67, 68, 69, 71, 72, 73, 74].

4 Synchronisation

Our incursion in this area dates back to the mid 2000s, in the context of switched synchronisation of mechanical systems, chaotic systems –[9] and stretches until recent work on nonlinear oscillators, in the context of neurosciences (see Part I of this document).

Publications on the subject of synchronisation in automatic control fora continues to grow exponentially fast and yet, it has been a centre of attention in several disciplines before control theory: it was introduced in vibration mechanics by Prof. Blekhman in the 1970s in USSR, and is increasingly popular among physicists in the context of synchronisation of chaotic systems since the early 1990s.

Controlled synchronisation of dynamical systems consists, generally speaking, in making two or more systems exhibit similar dynamical behaviour; this can be formulated with respect to the respective systems' outputs or states. In the first case, the problem is reminiscent of output regulation while in the second, it reminds us of tracking control. Recent applications in controlled synchronisation involve different areas of science and technology and include problems such as synchronisation of networks of nonlinear oscillators, formation control of vehicles, a problem that may be re-casted as a form of mutual synchronisation and which consists on having a set of autonomous vehicles - aerial, terrestrial or marine - advance in a coordinated formation to carry out a common mission. In this direction it is worth mentioning, *e.g.*, the problem of formation of satellites and of mobile robots. Other areas where the synchronisation problem covers importance include the control of mechanical systems, vibration-mechanics-based technologies and encoding of information for secure transmission, to mention a few. A more recent application is electrical stimulation and Deep-Brain stimulation –see Section 5 on p. 104.

4.1 Switched synchronisation

A particular trend of our work on switched controlled systems is synchronisation. From a control theory viewpoint, it has been established that master-slave synchronisation may be re-casted in an observer-design problem while controlled synchronisation may be addressed under appropriate assumptions as a classical tracking control problem. Other aspects classical in control theory, such as parameter uncertainty, robustness, optimality, *etc.*, arise naturally. In some cases, as for example in cooperative coordination of mobile robots, satellites or robot manipulators the controlled synchronisation problem includes two subtasks : tracking of a desired trajectory that is common to all robots and, second, the synchronisation of robots behaviour relative to each other. Strictly speaking only the second problem is about synchronisation.

We have studied and solved the problems of tracking and synchronisation, simultaneously; that is, we address the control problem of making a master system follow a reference desired trajectory and to make a set of slave systems synchronise with the master. We have showed that, in certain cases, the tasks of tracking, master-slave and mutual synchronisation are equivalent, up to an invertible mapping. In other words, such state transformation introduces a gain relation that may be significant when the systems are affected by external disturbances or in the presence of neglected dynamics. Our control approach to synchronisation is novel in the sense that it is based on a supervisor (along the lines discussed above) which switches between a tracking and a synchronisation controller, depending on the respective errors. In the particular case of mechanical systems we show that this results in a significant improvement of performance which may not be achieved following "classic" solutions to synchronisation problems. See [72, 73, 71].

4.2 Observer-based synchronisation

We solved the problem of master-slave synchronisation for a class of systems which include chaotic oscillators under parametric and state estimation using adaptive observers which cover high-gain based designs among others. Our results rely on theorems on stability for non-autonomous systems constructed based on conditions of persistency of excitation along trajectories –see Section 2.3 on p. 96. The class of nonlinear systems that fit our framework includes systems that are linear in the unknown variables but parametric uncertainty may appear anywhere in the model. The conditions of our adaptive observers intersect (and generalise in certain ways) with high-gain designs for nonlinear systems however, in contrast to other works on the subject our method is not restricted to high-gain observers. Also, the systems that we consider contain time-varying nonlinearities which may be regarded as neglected dynamics; this includes a variety of mechanical systems and chaotic oscillators.

We have proved that under relaxed assumptions (in terms of persistency of excitation and structural conditions such as detectability and observability) the synchronisation and parametric errors converge to compact sets, that is, the errors are bounded and relatively small (this is called practical asymptotic stability). Under more stringent conditions, for instance, in the case that parameters are known but not the states, synchronisation may be achieved. Similarly, the adaptive observers may be employed into the specific problem of parameter identification if full measurement of master states is available. This situation is similar to the context of tracking control with full state-feedback as mentioned earlier. Robust observer-based synchronisation is particularly well-suited for applications in secured transmission of information. In that context, the master is regarded as a transmitter, the slave as a receiver and the information plays a role of "perturbation". Precision of information recovery may be assimilated to a problem of robustness with respect to perturbations. See [76, 9].

4.3 Spacecraft formation control

Spacecraft formation control is a relatively new and active field of research. Formations, characterised by the ability to maintain relative positions without real-time ground commands, are motivated by the aim of placing measuring equipment further apart than what is possible on a single spacecraft. From a control design perspective, a crucial challenge is to maintain a predefined relative trajectory, even in the presence of disturbances. Most of these disturbances are hard to model in a precise manner. Only statistical or averaged characteristics of the perturbing signals (*e.g.*, amplitude, energy, average energy, *etc.*) are typically available.

Nonlinear control theory provides neat instruments to guarantee a prescribed precision in spite of these disturbances. However they mostly require that the amplitude, or the energy, of the perturbations be finite in order to ensure robustness and guaranteed estimates on the performance are "proportional" to the disturbing signal. In this context, we formulated new results that give hard bounds on the state norm for input-to-state stable systems in presence of signals with possibly unbounded amplitude and/or energy. Also, we enlarged the class of signals to which ISS systems are robust, by simply conducting a tighter analysis on these systems. In contrast to most previous works on input-to-state stability, the class of disturbances that we considered is defined on the basis of a sliding average. Then, we applied this new analysis result to the control of spacecraft formations. To this end, we exploited the Lyapunov function available for such systems to identify the class of signals to which the formation is robust. See [133, 33, 61].

4.4 Synchronisation and control of surface vessels

During our repeated visits to the Norwegian University of Science and Technology (NTNU) in Trondheim, Norway, since 1997, we have applied our expertise on cascaded systems theory in a collaboration with Prof. Thor I. Fossen, to the design of a controller plus observer scheme to solve the open problem of global dynamic positioning of nonlinear ships. It must be underlined that this control problem is a particular case of an under-actuated mechanic system due to the environmental disturbances. The difficulty is increased by the fact that only *noisy* position measurements are available [20]. See also [6].

In the case of surface marine vessels we may cite several problems on which we have worked in collaboration with our colleagues from NTNU such as cross-track formation control problem, "follow the leader" configuration, global dynamic positioning of nonlinear ships.

In the problem of so-called cross-track formation control, given a desired path, a formation pattern and a speed profile, the objective is to control a group of vessels so that they adopt, asymptotically, the desired formation move along the given path with the desired speed. Our experience on stability of nonlinear systems in cascade allowed to obtain excellent synchronisation results of the vehicles without any assumptions on the vehicles' dynamics, which are typically imposed in this case, and with arbitrarily connected topology of communication schemes.

In the case of surface marine vessels we may also cite the problem of formation of two ships in a "follow the leader" configuration; this is the case of underway replenishment that is, refuelling of a ship via a container source vessel while travelling in the open ocean. We addressed this problem using a "virtual vehicle" approach and, as a result, global practical stability of the system was established.

4.5 Tele-operation of mechanical systems

In the context of mechanical systems we also considered control design to ensure consensus of a network of non-identical Euler-Lagrange systems with variable time-delays in the communications. The interconnection network is modelled as an undirected weighted graph. Synchronisation of networks composed by fully-actuated robot manipulators was considered in [57], while the case of under-actuated robots was considered in [2]. Experimental evidence, using three fully-actuated 3-degrees-of-freedom robot manipulators, interconnected through the Internet, is included in [57] in order to support the theoretical results of [57].

4.6 Synchronisation and self-organised motion

In the past few years, we have been developing a new theory of consensus which generalises in different ways the actual paradigm. We consider heterogeneous networks of systems under diffusive coupling. That is, we assume that a set of systems with different dynamics (in general, different models and parameters) interact over a network.

We have established that, for the purpose of analysis, the behaviour of the systems interconnected over the network via diffusive coupling may be studied via two separate properties: the stability of what we call the emergent dynamics and the synchronisation errors of each of the units' motions, relative to an averaged system, also called "mean-field" system. The emergent dynamics is an averaged model of the systems' dynamics regardless of the inputs while the mean-field oscillator's motion corresponds to the average of the units' motions and its "steady-state" corresponds to the motion defined by the emergent dynamics. For instance, in the classical paradigm of consensus of a collection of integrators, which is a particular case of our framework, the emergent dynamics is null while the mean field trajectory corresponds to a weighted average of the states. For a balanced graph, we know that all units reach consensus and the steady-state value is an equilibrium point corresponding to the average of the initial conditions.

In our framework, the emergent dynamics possesses a stable attractor, in contrast to (the particular case of) an equilibrium point as is the case of the chain of integrators. Then, we say that the network presents dynamic consensus if there exists an attractor $\mathcal{A}$, in the phase-space of the emergent state, such that the trajectories of all units are attracted to $\mathcal{A}$ asymptotically and remain close to it. This, however, is possible only for homogeneous networks. In the setting of heterogeneous networks, only practical synchronisation is achievable in general that is, the trajectories of all units converge to a neighbourhood of the attractor of the emergent dynamics and remain close to this neighbourhood. Our tools give a qualitative but formal description of practical synchronisation.

This work is the most recent and, even though it is at its "*debut*" it has already led to the completion of two PhD theses recently defended entirely under our supervision –see farther below. The framework is also described in Part I of this document.

5 Neurosciences

One of the standard therapies for patients with Parkinson's disease is deep brain stimulation (DBS), which uses permanent electrodes planted in sub-cortical target areas to counteract a pathologically created synchronisation in a nerve cells population. In current clinical practice open-loop control is used for such stimulation and the parameters of stimulation (frequency, amplitude, pulse width, waveform, and size of the stimulating electrode) are chosen empirically. Our main objective is to provide a more adequate stimulation signal by using a synthetic model-based approach for control design, using instantaneous information available on the brain activity (average local field potential); this approach is expected to be much less invasive than the presently implemented techniques.

Recent theoretical work shows that the pathological neuronal activity is linked with the change of the average frequency of synchronisation of the corresponding group of neurons (local field potential). Therefore, the main objective of DBS treatment can be viewed as a change in the frequency of oscillation or even *desynchronisation*. However, the measurements of the neuronal activity available for DBS treatment are limited and noisy, which explains why this information is not exploited in the current practice - even the most recent techniques make use of a permanently exciting electrical stimulation (open-loop control), disregarding the evolution of the neuronal activity of the patient.

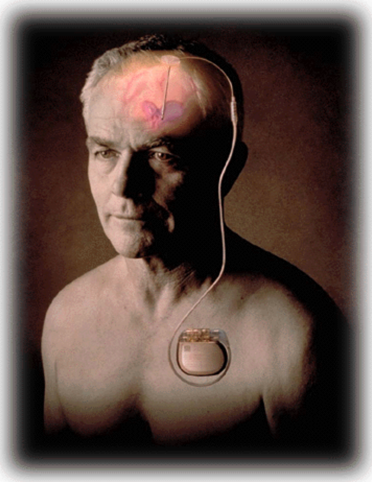

Figure 4.4: Deep-brain stimulation

Thus, there is an authentic need of formal analysis that lead to realistic yet, mathematically tractable models and, consequently, to closed-loop control techniques that are less invasive.

5.1 On (analysis of) neuronal networks models

At the microscopic level, neurons can be described as non-linear oscillators; then, neuronal networks ($e.g.$, cerebral structures) are regarded as built from the interconnection of a relatively large number of neurons. For a start, we have used Kuramoto's model to describe the behaviour of a neuronal network. It goes without saying that this model does not capture *all* of the effects inherent to the neuronal network, however, it does capture its intrinsic quality, $e.g.$, under certain conditions on the strength of interconnections, the synchronisation frequency of the neurons.

- **The thesis of L. Conteville** was devoted to the analysis of synchronisation of networks of neurons, notably, synchronisation issues related to Kuramoto's model. Firstly, a classical "all-to-all" Kuramoto's equation was used to model a simplified version of the neuronal network. Then, in order to analyse the synchronisation properties of the latter an auxiliary linear model was constructed that preserves information on the natural frequencies and interconnection gains of the neurons. The stability properties for the so-obtained models were analysed and we showed that, asymptotically, this system maps onto the original Kuramoto's model.

As a second approach, a more complex model of a neuronal network was adapted: to describe dynamic behaviour of the individual neurons we use the more representative Hindmarsh Rose model which, depending on the choice of the model parameters, provides a good qualitative description of the principal neuronal patterns: spiking, bursting and post-inhibitory rebound. We considered a heterogeneous network of diffusively coupled Hindmarsh Rose neurons and address several synchronisation issues for this network, notably, stability of the synchronisation and characterisation of the limiting synchronised behaviour. Difficulties and challenges in analysis of neuronal networks arise; on one hand, due to the complexity of oscillatory activity of a single neuronal model and, on the other, from the appearance of multi-stability for some parameter values.

See [42, 49, 52, 53, 54].

- **The thesis of A. El Ati.** While in the thesis of L. Conteville the interconnections among the neurons are considered to be undirected (symmetric Laplacian graph), in the thesis of A. El Ati this rather restrictive assumption is dropped so new questions are addressed which stem precisely from the non-symmetry of the interconnections. For instance, for Kuramoto's model even the frequency of synchronisation becomes a function of the interconnection gain, while for the symmetric case the frequency of synchronisation is simply an average of natural frequencies of neurons. Furthermore, some of the results on Kuramoto's model were extended to a more complicated model of nonlinear oscillators, that of Stuart-Landau oscillators. See [44, 48, 50, 55].

- **The post-doc of I. Haidar**, co-advised with W. Pasillas-Lepine and A. Chaillet, was devoted to the study of the neuronal system on a mesoscopic level: several cerebral structures are critically involved in the generation of abnormal oscillations in basal ganglia related to PD: the subthalamic nucleus (STN), primary motor cortex (M1), thalamus and globus pallidus (GPe) with the objective to analyse the simplified models, each of them summarising the macroscopic activity of a specific area and functional interconnections in between different areas. There are multiple problems of analysis of such neuronal models: adequate modelling of neuronal populations based on models of individual neurons, uncertain parameters in characterisation of interconnections between the populations, the non-symmetric structure of interconnections and delays in the latter, to name a few.

At mesoscopic level we follow a similar line of research and, in addition, we address issues related to stability of the oscillations in basal ganglia and effects induced by its interconnection with other zones. We analyse the simplified models, each of which summarises the macroscopic activity of a specific area and functional interconnections between different areas. We expect that such a model will serve later as the basis for development of new control strategies. See [3, 46, 51].

5.2 For a better treatment of neurological pathologies

Basal ganglia are interconnected deep brain structures involved in the generation of movement. Their persistent beta-band oscillations (13-30Hz) are known to be linked to Parkinson's disease motor symptoms. We have established conditions under which these oscillations may occur, by explicitly considering the role of the pedunculo-pontine nucleus (PPN). We analysed the existence of equilibria in the associated firing-rate dynamics and study their stability by relying on a delayed multi-input multi-output frequency analysis. We found that the PPN has an influence on the generation of pathological beta-band oscillations.

We developed a new closed-loop strategy for deep brain stimulation, derived using a model-based analysis of the basal ganglia. The system is described using a firing-rate model that has been proposed recently in the literature, in order to analyse the generation of beta-band oscillations. On this system, a proportional regulation of the firing-rate compensates the loss of stability of the subthalamo-pallidal loop in the Parkinsonian case. Nevertheless, because of actuation and measurement delays in the stimulation device, a filter with a well chosen frequency must be added to such proportional schemes, in order to achieve an adequate stability margin.

- **Desynchronisation.** As we have explained, there exists theoretical evidence that synchronisation of firing rates is directly linked to the appearance of neurological pathologies.

We introduced two notions of *de*synchronisation for interconnected phase oscillators by requiring that phases drift away from one another either at all times or in average. We provided a characterisation of each of these two notions based on the grounded variable associated to the system, and relates them to a classical notion of instability valid in Euclidean spaces. An illustration is provided through the Kuramoto system, which is shown to be desynchronisable by proportional mean-field feedback.

In terms of automatic control we can formulate this problem as (de)synchronisation of a large network of interconnected neurons using a single sampled measured and a single sampled control input - sampling is due to the fact that the same electrode is used both for measurements and stimulation. See [5, 56, 60].

List of publications

A. Journal papers

1. L. Greco, A. Chaillet, and E. Panteley, "Robustness of stochastic discrete-time switched linear systems with application to control with shared resources," *Trans. Automat. Control*, 2014. Accepté.
2. S. Avila-Becerril, G. Espinosa-Perez, E. Panteley, and R. Ortega, "Consensus control of flexible joint robots," Submitted to *Int. J. Control*, 2013. Accepté.
3. I. Haidar, W. Pasillas-Lépine, E. Panteley, A. Chaillet, S. Senova, and S. Palfi, "Analysis of delay-induced basal ganglia oscillations: the role of external excitory nuclei," *Int. J. Control*, vol. 87, no. 9, pp. 1936–1956, 2014. DOI: 10.1080/00207179.2014.895925.
4. R. Ortega and E. Panteley, "Comments on "$\mathcal{L}_1$-adaptive control: stabilization mechanism, existing conditions for stability and performance limitations"," *Int. J. Control*, 2013. online 30.10.2013.
5. A. Franci, A. Chaillet, E. Panteley, and F. Lamnabhi-lagarrigue, "Desynchronization and inhibition of kuramoto oscillators by scalar mean-field feedback," *Mathematics of Control, Signals, and Systems*, vol. 24, no. 1-2, pp. 167–217, 2012.
6. E. Børhaug, A. Pavlov, E. Panteley, , and K. Y. Pettersen, "Straight line path following for formations of underactuated marine surface vessels," *IEEE Trans. on Control Systems Technology*, vol. 19, no. 13, pp. 493–506, 2011.
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